Secondary simplex method for 2-Stage Stochastic Linear Problem

Maël Forcier, Stéphane Gaubert

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SMAI MODE

mael.forcier@enpc.fr

Contents

Linear Programming

$$\min_{x \in \mathbb{R}^n} \quad c^\top x$$
$$\text{s.t.} \quad Ax \leqslant b$$

Example: $P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$

$$A = \begin{pmatrix} 1 & 1 \\ & \\ & \\ & \\ & \\ & \\ & \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ \\ \\ \\ \\ \\ \end{pmatrix}$$

$x_1 + x_2 \leqslant 1$ (1)

(2)

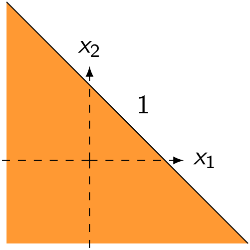
(3)

(4)

(5)

(6)

(7)



Linear Programming

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & c^\top x \\ \text{s.t.} \quad & Ax \leqslant b \end{aligned}$$

Example: $P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ & \\ & \\ & \\ & \\ & \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 1 \\ \\ \\ \\ \\ \\ \end{pmatrix}$$

$x_1 + x_2 \leqslant 1$ (1)

$x_1 - x_2 \leqslant 1$ (2)

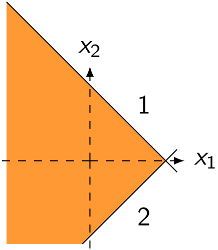
(3)

(4)

(5)

(6)

(7)



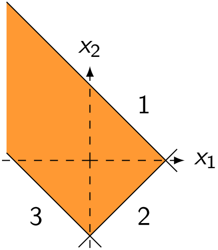
Linear Programming

$$\min_{x \in \mathbb{R}^n} \quad c^\top x$$
$$\text{s.t.} \quad Ax \leqslant b$$

Example: $P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -1 & -1 \\ & \\ & \\ & \\ & \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \\ \\ \\ \end{pmatrix}$$

$$x_1 + x_2 \leqslant 1 \qquad (1)$$
$$x_1 - x_2 \leqslant 1 \qquad (2)$$
$$-x_1 - x_2 \leqslant 1 \qquad (3)$$
$$(4)$$
$$(5)$$
$$(6)$$
$$(7)$$



Linear Programming

$$\min_{x \in \mathbb{R}^n} \quad c^\top x$$
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Example: $P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -1 & -1 \\ -1 & 1 \\ & \\ & \\ & \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ \\ \\ \\ \end{pmatrix}$$

$$x_1 + x_2 \leqslant 1 \quad (1)$$
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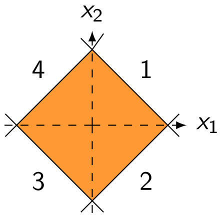


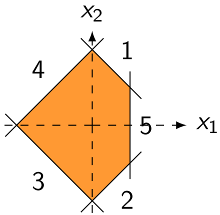
Linear Programming

$$\min_{x\in\mathbb{R}^n} \quad c^\top x$$
$$\text{s.t.} \quad Ax \leqslant b$$

Example: $P = \{x \in \mathbb{R}^n \,|\, Ax \leqslant b\}$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -1 & -1 \\ -1 & 1 \\ 1 & 0 \\ & \\ & \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0.5 \\ \\ \end{pmatrix}$$

$$\begin{aligned} x_1 + x_2 &\leqslant 1 && (1)\\ x_1 - x_2 &\leqslant 1 && (2)\\ -x_1 - x_2 &\leqslant 1 && (3)\\ -x_1 + x_2 &\leqslant 1 && (4)\\ x_1 &\leqslant 0.5 && (5)\\ & && (6)\\ & && (7) \end{aligned}$$



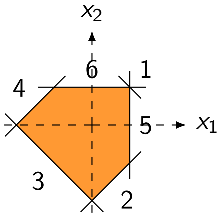
Linear Programming

$$\min_{x \in \mathbb{R}^n} \quad c^\top x$$
$$\text{s.t.} \quad Ax \leqslant b$$

Example: $P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -1 & -1 \\ -1 & 1 \\ 1 & 0 \\ 0 & 1 \\ & \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0.5 \\ 0.5 \\ \end{pmatrix}$$

$$\begin{aligned} x_1 + x_2 &\leqslant 1 && (1) \\ x_1 - x_2 &\leqslant 1 && (2) \\ -x_1 - x_2 &\leqslant 1 && (3) \\ -x_1 + x_2 &\leqslant 1 && (4) \\ x_1 &\leqslant 0.5 && (5) \\ x_2 &\leqslant 0.5 && (6) \\ & && (7) \end{aligned}$$



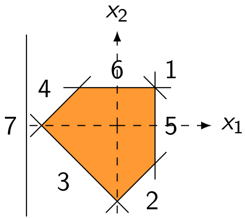
Linear Programming

$$\min_{x \in \mathbb{R}^n} \quad c^\top x$$
$$\text{s.t.} \quad Ax \leqslant b$$

Example: $P = \{x \in \mathbb{R}^n \,|\, Ax \leqslant b\}$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -1 & -1 \\ -1 & 1 \\ 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0.5 \\ 0.5 \\ -1.2 \end{pmatrix}$$

$$\begin{aligned} x_1 + x_2 &\leqslant 1 && (1) \\ x_1 - x_2 &\leqslant 1 && (2) \\ -x_1 - x_2 &\leqslant 1 && (3) \\ -x_1 + x_2 &\leqslant 1 && (4) \\ x_1 &\leqslant 0.5 && (5) \\ x_2 &\leqslant 0.5 && (6) \\ x_1 &\geqslant -1.2 && (7) \end{aligned}$$



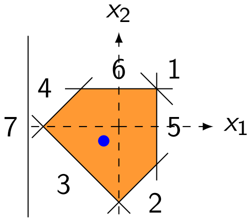
Active constraints

Definition

We denote by $\mathcal{I}(A, b)$, the collection of sets of active constraints as :

$$\mathcal{I}(A, b) = \{I_{A,b}(x) \mid Ax \leqslant b\}$$

with $I_{A,b}(x) := \{i \in [q] \mid A_i x = b_i\}$



$P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$

$I_{A,b}(x) = \emptyset$

To ease the notation, we write:

$$\mathcal{I}(A, b) = \{\emptyset, \qquad \}$$

Active constraints

Definition

We denote by $\mathcal{I}(A, b)$*, the collection of sets of active constraints as :*

$$\mathcal{I}(A, b) = \{I_{A,b}(x) \mid Ax \leqslant b\}$$

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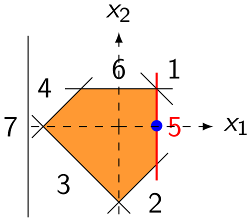

$P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$

$I_{A,b}(x) = \{5\}$

To ease the notation, we write:

$\mathcal{I}(A, b) = \{\emptyset, 5, \qquad \}$

Active constraints

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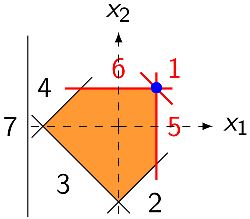
with $I_{A,b}(x) := \{i \in [q] \mid A_i x = b_i\}$



$P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$

$I_{A,b}(x) = \{1, 5, 6\}$

To ease the notation, we write:

$\mathcal{I}(A,b) = \{\emptyset, 5, 156, \qquad \}$

Active constraints

Definition

We denote by $\mathcal{I}(A,b)$*, the collection of sets of active constraints as :*

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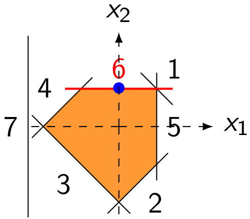
with $I_{A,b}(x) := \{i \in [q] \mid A_i x = b_i\}$



$P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$

$I_{A,b}(x) = \{6\}$

To ease the notation, we write:

$$\mathcal{I}(A,b) = \{\emptyset, 5, 156, 6, \qquad \}$$

Active constraints

Definition

We denote by $\mathcal{I}(A, b)$, the collection of sets of active constraints as :

$$\mathcal{I}(A, b) = \{I_{A,b}(x) \mid Ax \leqslant b\}$$

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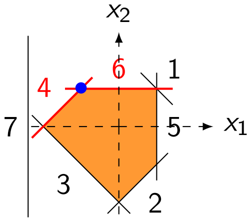

$P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$

$I_{A,b}(x) = \{4, 6\}$

To ease the notation, we write:

$$\mathcal{I}(A, b) = \{\emptyset, 5, 156, 6, 46, \qquad \}$$

Active constraints

Definition

We denote by $\mathcal{I}(A, b)$, the collection of sets of active constraints as :

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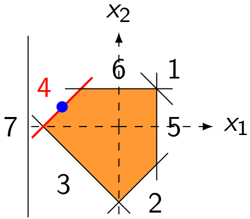

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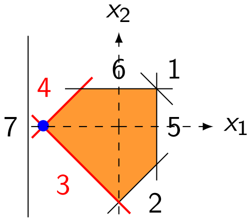

$P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$

$I_{A,b}(x) = \{3, 4\}$

To ease the notation, we write:

$$\mathcal{I}(A, b) = \{\emptyset, 5, 156, 6, 46, 4, 34, \qquad \}$$

Active constraints

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We denote by $\mathcal{I}(A, b)$, the collection of sets of active constraints as :

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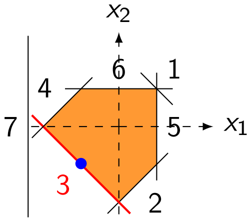
with $I_{A,b}(x) := \{i \in [q] \mid A_i x = b_i\}$



$P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$

$I_{A,b}(x) = \{3\}$

To ease the notation, we write:

$$\mathcal{I}(A, b) = \{\emptyset, 5, 156, 6, 46, 4, 34, 3, \qquad \}$$

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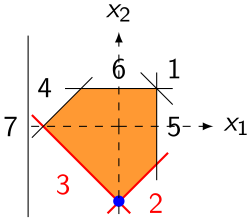
with $I_{A,b}(x) := \{i \in [q] \mid A_i x = b_i\}$



$P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$

$I_{A,b}(x) = \{2, 3\}$

To ease the notation, we write:

$$\mathcal{I}(A, b) = \{\emptyset, 5, 156, 6, 46, 4, 34, 3, 23, \qquad\}$$

Active constraints

Definition

We denote by $\mathcal{I}(A, b)$, the collection of sets of active constraints as :

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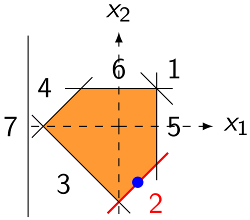

$P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$

$I_{A,b}(x) = \{2\}$

To ease the notation, we write:

$$\mathcal{I}(A, b) = \{\emptyset, 5, 156, 6, 46, 4, 34, 3, 23, 2, \quad \}$$

Active constraints

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We denote by $\mathcal{I}(A, b)$, the collection of sets of active constraints as :

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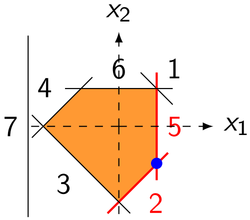
with $I_{A,b}(x) := \{i \in [q] \mid A_i x = b_i\}$



$P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$

$I_{A,b}(x) = \{2, 5\}$

To ease the notation, we write:

$$\mathcal{I}(A, b) = \big\{\emptyset, 5, 156, 6, 46, 4, 34, 3, 23, 2, 25\big\}$$

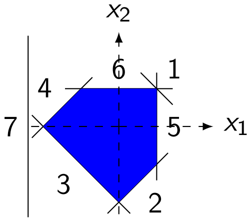
Faces

Definition

Let $I \in \mathcal{I}(A, b)$, we denote by P^I the face of P such that:

$$P^I = \{x \in P \mid A_I x = b_I\}$$

We have $\dim(P^I) = n - \text{rg}(A_I)$
Example for $I = \emptyset$



P, P^I

Faces

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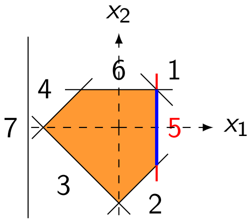
Example for $I = \{5\}$



P, P^I

Faces

Definition

Let $I \in \mathcal{I}(A, b)$, we denote by P^I the face of P such that:

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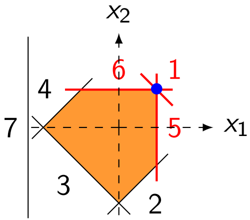
We have $\dim(P^I) = n - \mathrm{rg}(A_I)$
Example for $I = \{1, 5, 6\}$



P, P^I

Faces

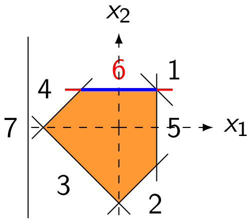
Definition

Let $I \in \mathcal{I}(A, b)$*, we denote by* P^I *the face of* P *such that:*

$$P^I = \{x \in P \mid A_I x = b_I\}$$

We have $\dim(P^I) = n - \mathrm{rg}(A_I)$

Example for $I = \{6\}$



P, P^I

Faces

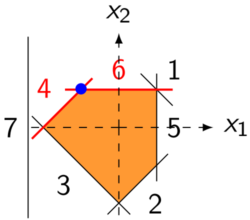
Definition

Let $I \in \mathcal{I}(A, b)$, we denote by P^I the face of P such that:

$$P^I = \{x \in P \mid A_I x = b_I\}$$

We have $\dim(P^I) = n - \mathrm{rg}(A_I)$

Example for $I = \{4, 6\}$



P, P^I

Faces

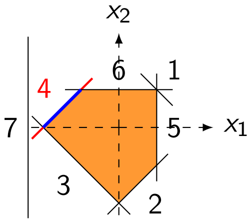
Definition

Let $I \in \mathcal{I}(A, b)$, we denote by P^I the face of P such that:

$$P^I = \{x \in P \mid A_I x = b_I\}$$

We have $\dim(P^I) = n - \mathrm{rg}(A_I)$

Example for $I = \{4\}$



P, P^I

Faces

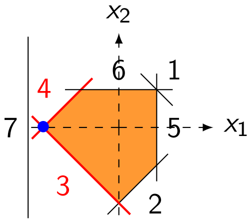
Definition

Let $I \in \mathcal{I}(A, b)$, we denote by P^I the face of P such that:

$$P^I = \{x \in P \mid A_I x = b_I\}$$

We have $\dim(P^I) = n - \mathrm{rg}(A_I)$

Example for $I = \{3, 4\}$



P, P^I

Faces

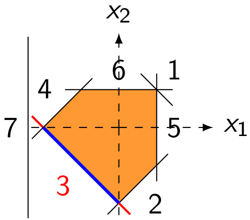
Definition

Let $I \in \mathcal{I}(A, b)$, we denote by P^I the face of P such that:

$$P^I = \{x \in P \mid A_I x = b_I\}$$

We have $\dim(P^I) = n - \mathrm{rg}(A_I)$

Example for $I = \{3\}$



P, P^I

Faces

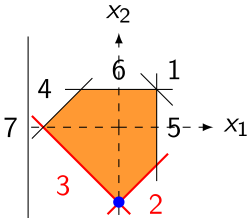
Definition

Let $I \in \mathcal{I}(A, b)$, we denote by P^I the face of P such that:

$$P^I = \{x \in P \mid A_I x = b_I\}$$

We have $\dim(P^I) = n - \text{rg}(A_I)$

Example for $I = \{2, 3\}$



P, P^I

Faces

Definition

Let $I \in \mathcal{I}(A, b)$, we denote by P^I the face of P such that:

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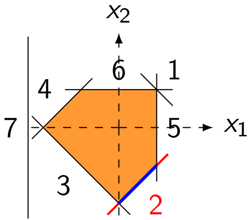
We have $\dim(P^I) = n - \mathrm{rg}(A_I)$

Example for $I = \{2\}$



P, P^I

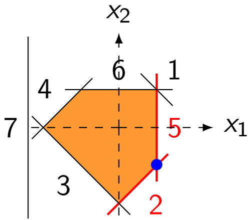
Faces

Definition

Let $I \in \mathcal{I}(A, b)$, we denote by P^I the face of P such that:

$$P^I = \{x \in P \mid A_I x = b_I\}$$

We have $\dim(P^I) = n - \mathrm{rg}(A_I)$
Example for $I = \{2, 5\}$

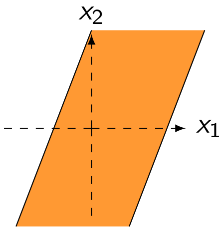


P, P^I

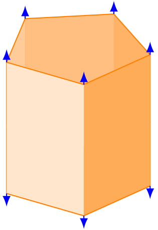
Lineality space, vertices and bases

Definition (Lineality space)

$\mathrm{Lin}(C) := \{u \in C \mid \forall t \in \mathbb{R},\ \forall x \in c,\ x + tu \in C\}.$



If
$P = \{x \in \mathbb{R}^n | Ax \leqslant b\}$,
then $\mathrm{Lin}(P) = \mathrm{Ker}(A)$



Definition (Bases and vertices)

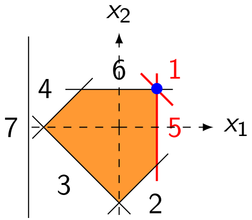
A basis B is a subset of $[p]$ such that $A_B = (A_{i,j})_{i \in B, 1 \leqslant j \leqslant n}$ is invertible.
A vertex of P is a face of dimension 0. $\mathrm{Vert}(P)$ is the set of vertices.

$\mathrm{Vert}(P) \neq \emptyset \iff A$ admits at least one basis $\iff \mathrm{rg}(A) = n \iff \mathrm{Lin}(P) = \{0\}$

We make this assumption without loss of generality.

Simplex method

Geometrically:
follow a path on the polyhedron from vertex to vertex

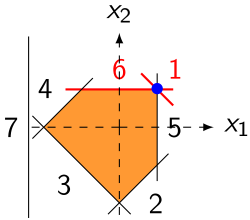


Combinatorially:
pivoting from basis to basis

$$B_1 = \{1, 5\}$$

Simplex method

Geometrically:
follow a path on the polyhedron from vertex to vertex

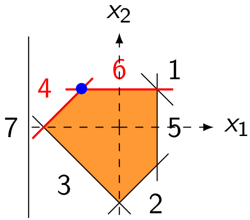


Combinatorially:
pivoting from basis to basis

$$B_1 = \{1, 5\}$$
$$B_2 = \{1, 6\}$$

Simplex method

Geometrically:
follow a path on the polyhedron from vertex to vertex



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$$B_1 = \{1, 5\}$$
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$$B_3 = \{4, 6\}$$

Simplex method

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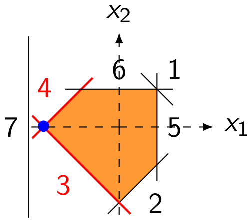

Combinatorially:
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$$B_1 = \{1,5\}$$
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$$B_2 = \{3,4\}$$

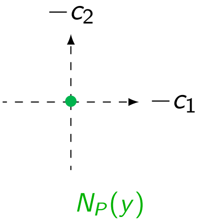
Normal fan $\mathcal{N}(P)$

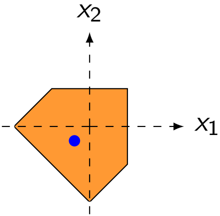
Definition

The normal fan of the polyhedron P is

$$\mathcal{N}(P) := \{N_P(x) \,|\, x \in P\}$$

with $N_P(x) = \{c \,|\, \forall x' \in P,\ c^\top(x' - x) \leqslant 0\}$ *the normal cone of P on x.*





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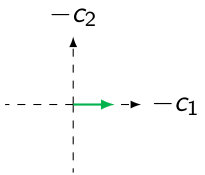

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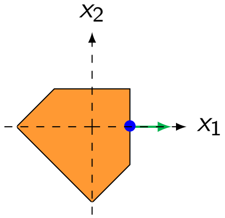

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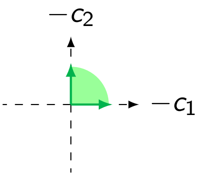

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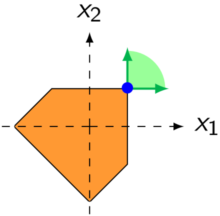

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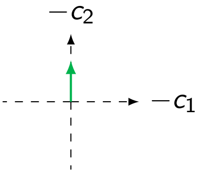

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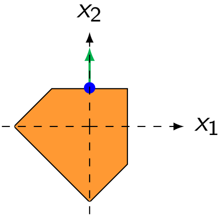

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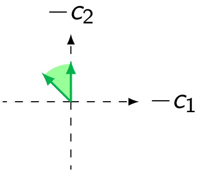

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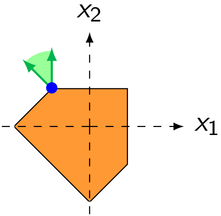

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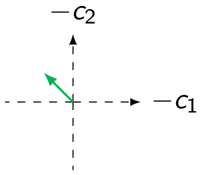

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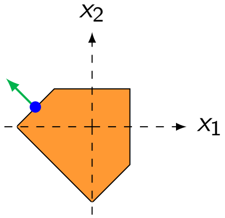

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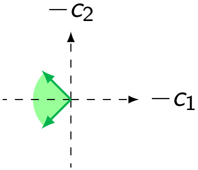

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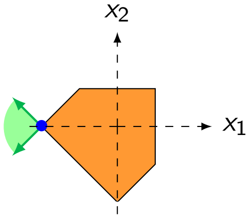

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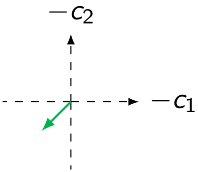

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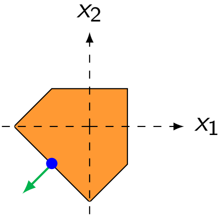

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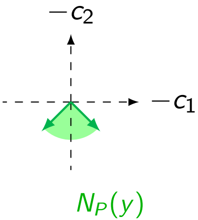

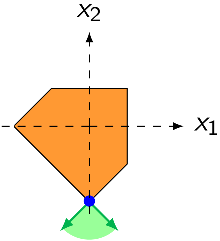

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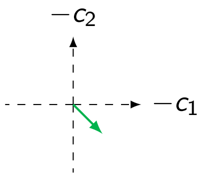

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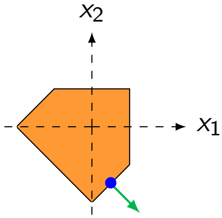

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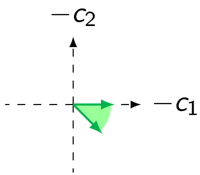

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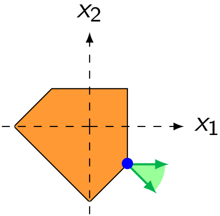

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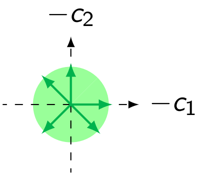
Proposition

$\{\mathrm{ri}(N) \mid N \in \mathcal{N}(P)\}$ *is a partition of* $\operatorname{supp} \mathcal{N}(P)$ $(= \mathbb{R}^m$ *if* P *is bounded).*



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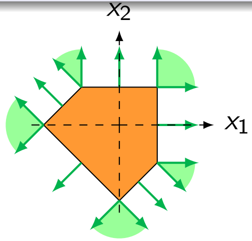


P and $\mathcal{N}(P)$

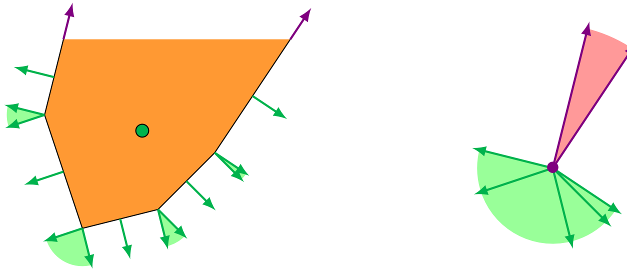
Definition (Recession cone)

$\mathrm{rc}(C) := \{u \in C \mid \forall t \in \mathbb{R}_+,\ \forall x \in c,\ x + tu \in C\}.$

Let $P = \{x \mid Ax \leqslant b\}$

$$\mathrm{rc}(P) = \{u \mid Au \leqslant 0\}$$

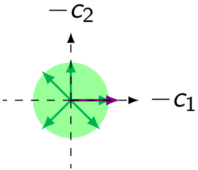
$$-\infty < \begin{cases} \inf_{x \in \mathbb{R}^n} & c^\top x \\ \text{s.t.} & Ax \leqslant b \end{cases} \iff -c \in \mathrm{rc}(P)^\star = \mathrm{Cone}(A^\top) = \mathrm{supp}\left(\mathcal{N}(P)\right)$$



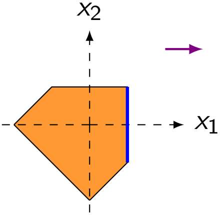
$\mathcal{N}(P)$: partition of cost coherent with the min

For any $N \in \mathcal{N}(P)$ and $-c \to \arg\min_{x \in P} c^\top x$ is constant for all $-c \in \mathrm{ri}(N)$.

$\arg\min_{x \in P} c^\top x$ is a face of P.



Cost $-c$ and $\mathcal{N}(P)$



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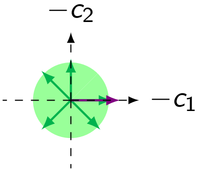

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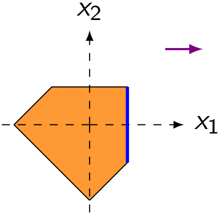

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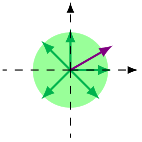

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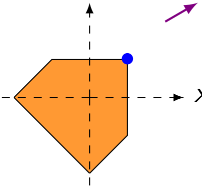

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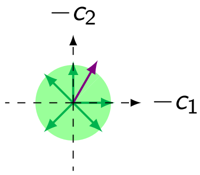

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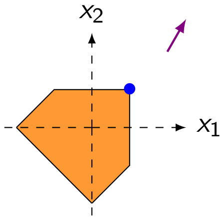

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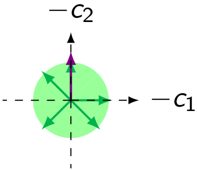

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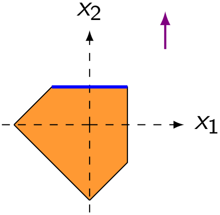

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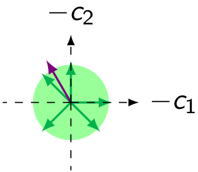

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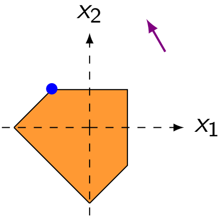

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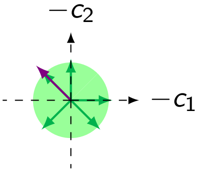

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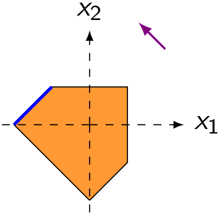

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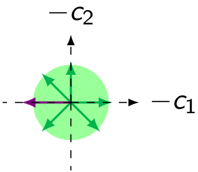

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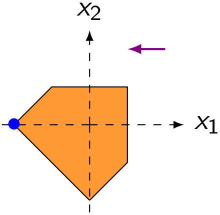

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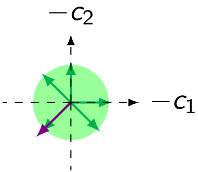

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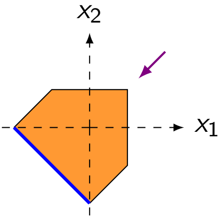

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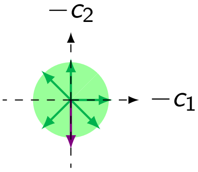

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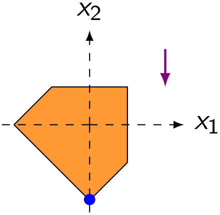

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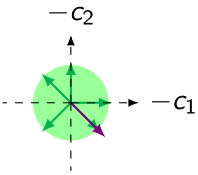
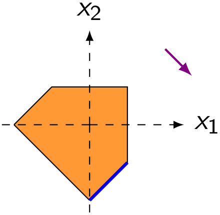


$\mathcal{N}(P)$

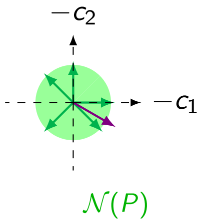


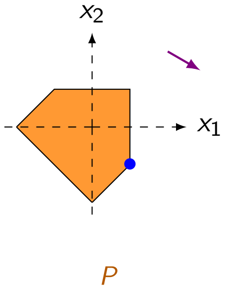
P

$\mathcal{N}(P)$: partition of cost coherent with the min

For any $N \in \mathcal{N}(P)$ and $-c \to \arg\min_{x \in P} c^\top x$ is constant for all $-c \in \text{ri}(N)$.

$\arg\min_{x \in P} c^\top x$ is a face of P.





$\mathcal{N}(P)$: partition of cost coherent with the min

For any $N \in \mathcal{N}(P)$ and $-c \to \arg\min_{x\in P} c^\top x$ is constant for all $-c \in \mathrm{ri}(N)$.

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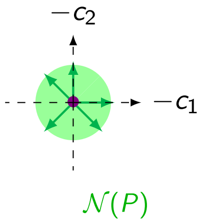
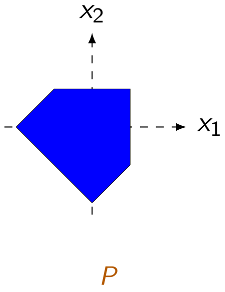
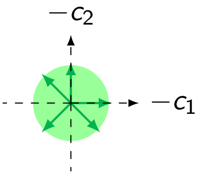




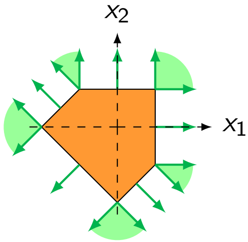
$\mathcal{N}(P)$: partition of cost coherent with the min

For any $N \in \mathcal{N}(P)$ and $-c \to \arg\min_{x \in P} c^\top x$ is constant for all $-c \in \mathrm{ri}(N)$.

$\arg\min_{x \in P} c^\top x$ is a face of P.

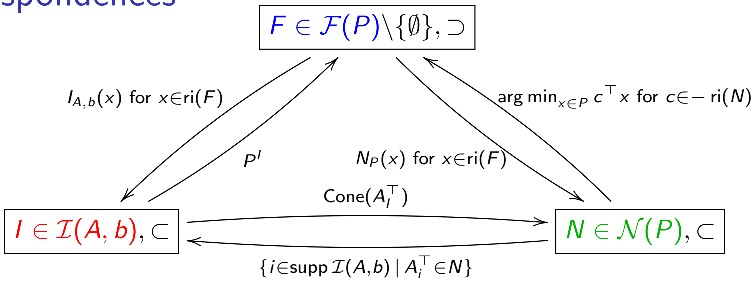


$\mathcal{N}(P)$



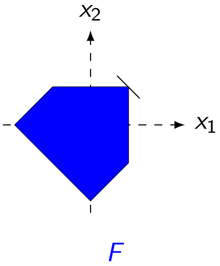
P

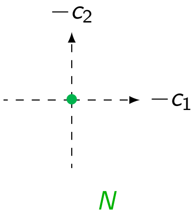
Correspondences



$$\mathrm{rg}(A_I) = n - \dim(F) = \dim(N)$$

$I = \emptyset$





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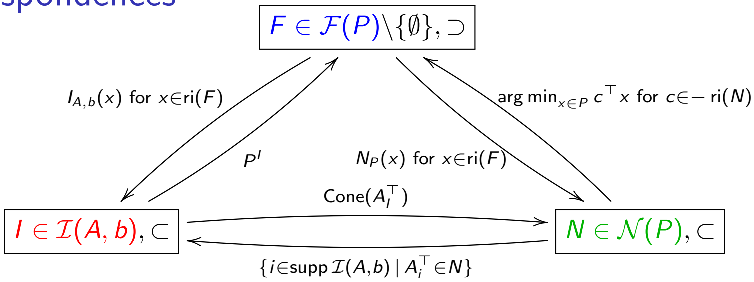
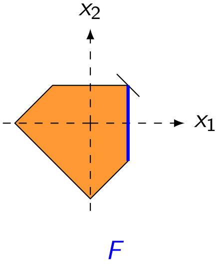
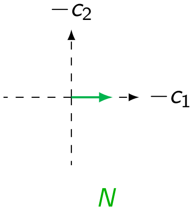


$$\mathrm{rg}(A_I) = n - \dim(F) = \dim(N)$$

$I = \{5\}$





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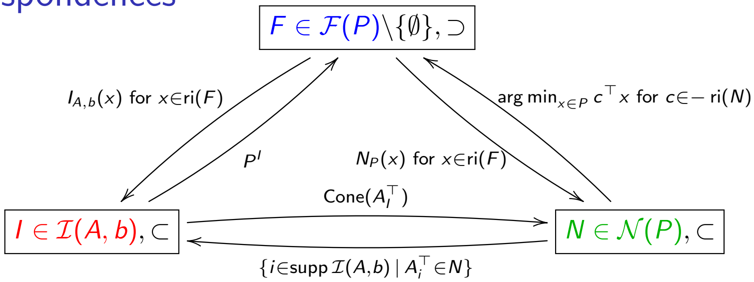
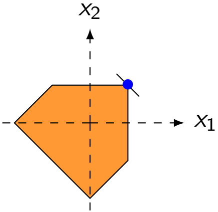
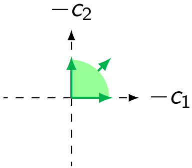


$$\mathrm{rg}(A_I) = n - \dim(F) = \dim(N)$$

$I = \{1, 5, 6\}$

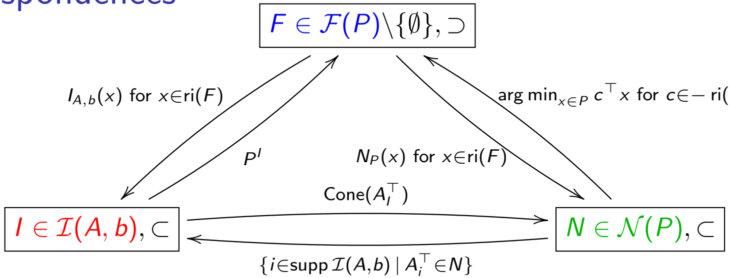


F



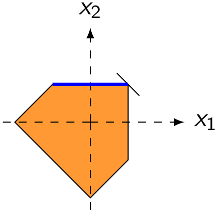
N

Correspondences



$$\mathrm{rg}(A_I) = n - \dim(F) = \dim(N)$$

$I = \{6\}$



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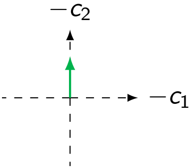

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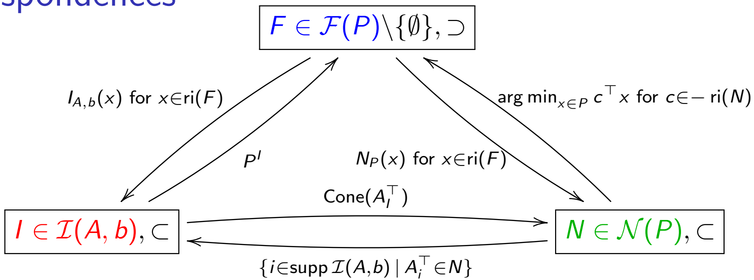
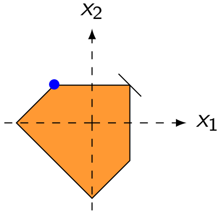


$$\mathrm{rg}(A_I) = n - \dim(F) = \dim(N)$$

$I = \{4, 6\}$



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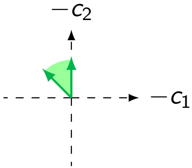

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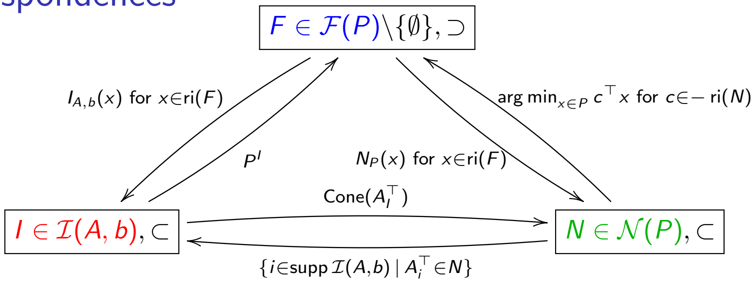
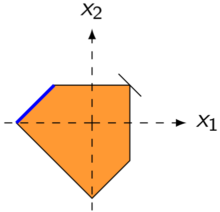


$$\mathrm{rg}(A_I) = n - \dim(F) = \dim(N)$$

$I = \{4\}$



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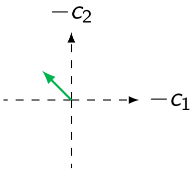

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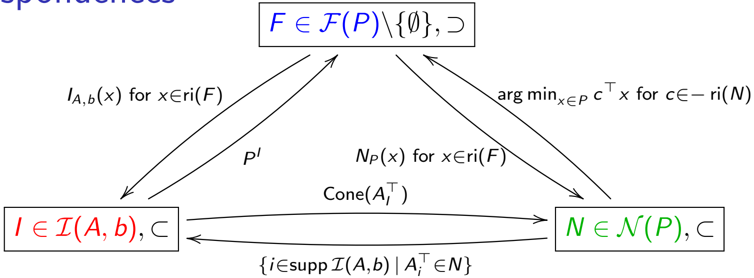
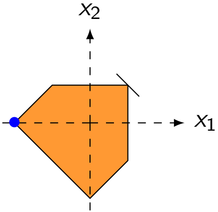


$$\mathrm{rg}(A_I) = n - \dim(F) = \dim(N)$$

$I = \{3, 4\}$



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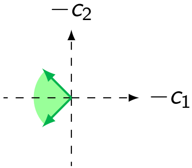

N

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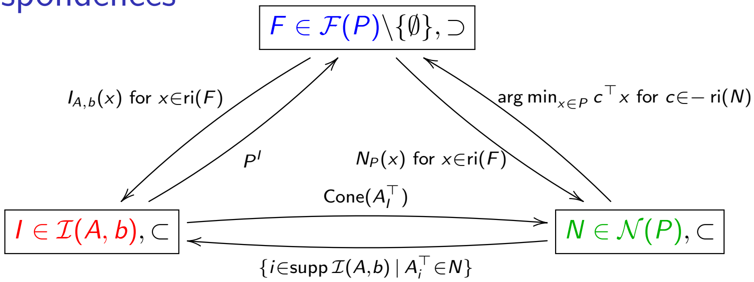
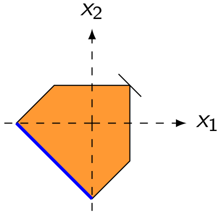


$$\mathrm{rg}(A_I) = n - \dim(F) = \dim(N)$$

$I = \{3\}$



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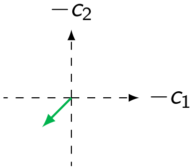

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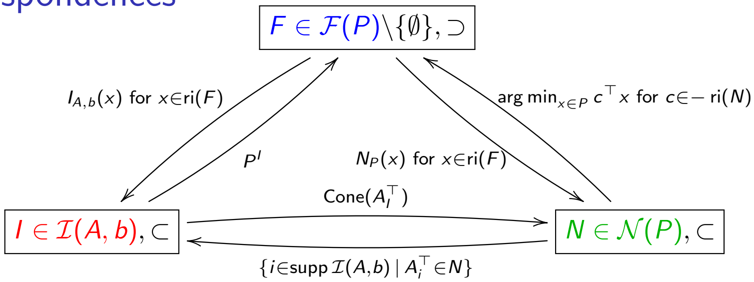
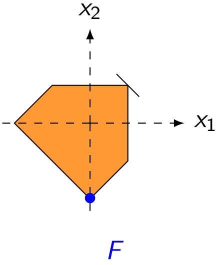
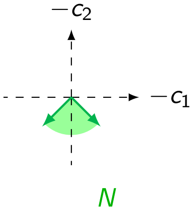


$$\mathrm{rg}(A_I) = n - \dim(F) = \dim(N)$$

$I = \{2, 3\}$





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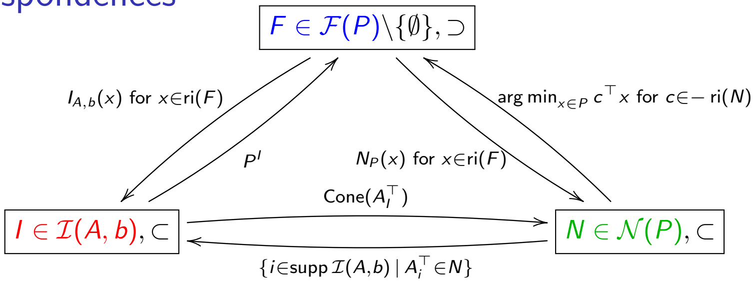
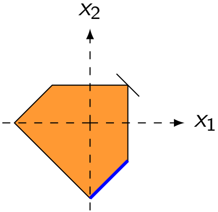


$$\mathrm{rg}(A_I) = n - \dim(F) = \dim(N)$$

$I = \{2\}$



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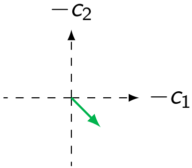

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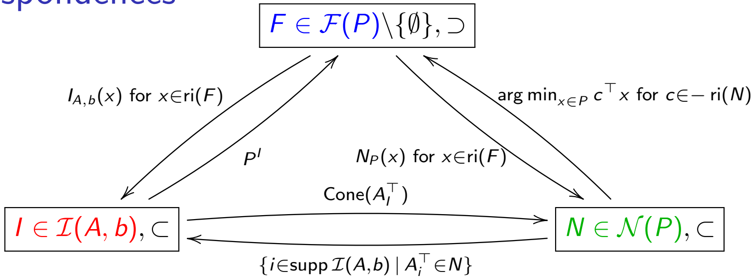
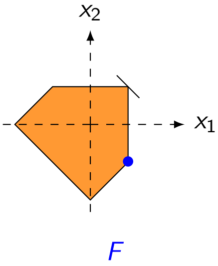
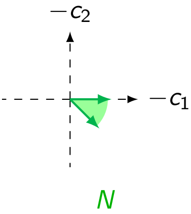


$$\mathrm{rg}(A_I) = n - \dim(F) = \dim(N)$$

$I = \{2, 5\}$





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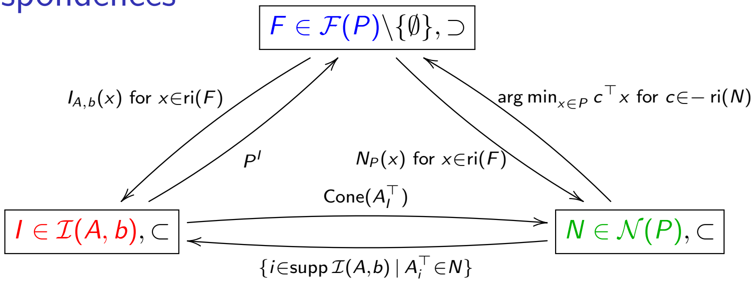
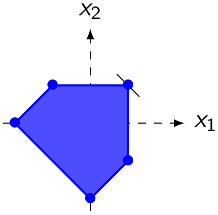
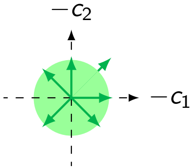


$$\mathrm{rg}(A_I) = n - \dim(F) = \dim(N)$$

$\mathcal{I}(A,b) =$
$\{\emptyset, 5, 156, 6, 46, 4,$
$34, 3, 23, 2, 25\}$

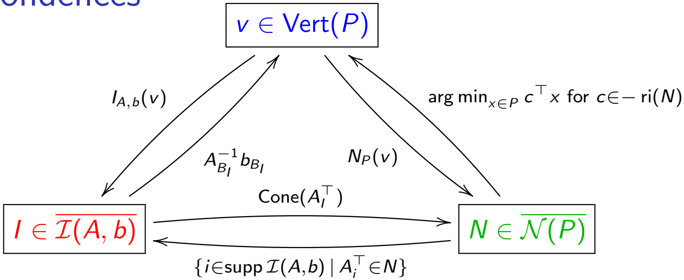


$\mathcal{F}(P)$

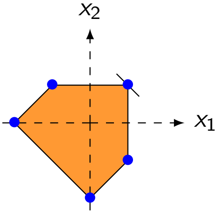


$\mathcal{N}(P)$

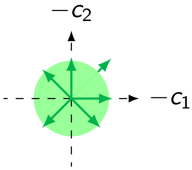
Correspondences



$\overline{\mathcal{I}(A, b)} =$
$\{156, 46, 34, 23, 25\}$



$\mathrm{Vert}(P)$



$\overline{\mathcal{N}(P)}$

Link with regular subdivisions

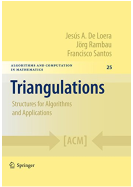
Definition (DLRS10)

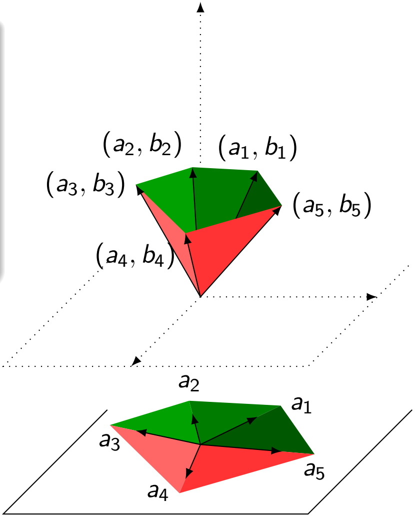
$$\mathcal{S}(A^\top, b) := \{I_F \mid F \in \mathcal{F}_{\text{low}}(LC_{A^\top,b})\}$$

$$LC_{A^\top,b} := \text{Cone}\left(\begin{pmatrix} a_i \\ b_i \end{pmatrix}_{i\in[q]}\right)$$

$$I_F := \{i \in [q] \mid (a_i, b_i) \in F\}.$$

$$\mathcal{S}(A^\top, b) = \mathcal{I}(A, b)$$





$$\mathcal{I}(W^\top q) = \mathcal{I}_{com} \cup \{\{5\}, \{4, 5\}, \{1, 5\}\}$$

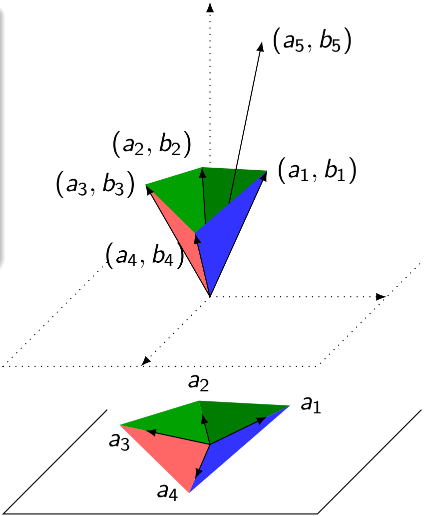
Link with regular subdivisions

Definition (DLRS10)

$$\mathcal{S}(A^\top, b) := \{I_F \mid F \in \mathcal{F}_{\text{low}}(LC_{A^\top,b})\}$$

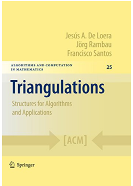
$$LC_{A^\top,b} := \text{Cone}\left(\begin{pmatrix} a_i \\ b_i \end{pmatrix}_{i\in[q]}\right)$$

$$I_F := \{i \in [q] \mid (a_i, b_i) \in F\}.$$



$$\mathcal{S}(A^\top, b) = \mathcal{I}(A, b)$$

$$\mathcal{I}(W^\top, q) = \mathcal{I}_{com} \cup \{\{1, 4\}\}$$



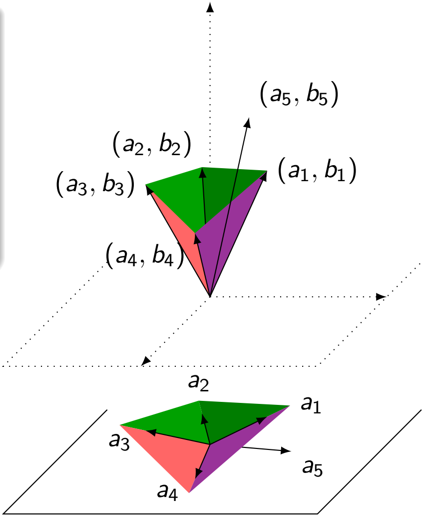
Link with regular subdivisions

Definition (DLRS10)

$$\mathcal{S}(A^\top, b) := \{I_F \mid F \in \mathcal{F}_{\text{low}}(LC_{A^\top,b})\}$$

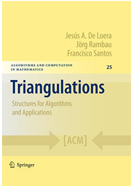
$$LC_{A^\top,b} := \text{Cone}\left(\begin{pmatrix} a_i \\ b_i \end{pmatrix}_{i \in [q]}\right)$$

$$I_F := \{i \in [q] \mid (a_i, b_i) \in F\}.$$



$$\mathcal{S}(A^\top, b) = \mathcal{I}(A, b)$$

$$\mathcal{I}(W^\top q) = \mathcal{I}_{com} \cup \{\{1, 4, 5\}\}$$



Contents

2-Stage Stochastic Linear Programming

$$\min_{x \in \mathbb{R}^n} \quad c^\top x + \mathbb{E}\begin{bmatrix} \min_{y \in \mathbb{R}^m} & \mathbf{q}^\top y \\ \text{s.t.} & Tx + Wy \leqslant h \\ & \end{bmatrix} \qquad \text{(2SLP)}$$
$$\text{s.t.} \quad Ax \leqslant b$$

where $T \in \mathbb{R}^{p \times n}, W \in \mathbb{R}^{p \times m}$ and $h \in \mathbb{R}^p$.
We can assume $A = 0$ and $b = 0$:
We set

$$\widetilde{T} := \begin{pmatrix} T \\ A \end{pmatrix}, \quad \widetilde{W} := \begin{pmatrix} W \\ 0 \end{pmatrix} \quad \text{and } \widetilde{h} = \begin{pmatrix} h \\ b \end{pmatrix}$$

2-Stage Stochastic Linear Programming

$$\min_{x \in \mathbb{R}^n} \quad c^\top x + \mathbb{E}\begin{bmatrix} \min\limits_{y \in \mathbb{R}^m} & \mathbf{q}^\top y & \\ \text{s.t.} & Tx + Wy \leqslant h \\ & Ax \qquad \leqslant b \end{bmatrix}$$

where $T \in \mathbb{R}^{p \times n}, W \in \mathbb{R}^{p \times m}$ and $h \in \mathbb{R}^p$.
We can assume $A = 0$ and $b = 0$:
We set

$$\widetilde{T} := \begin{pmatrix} T \\ A \end{pmatrix}, \quad \widetilde{W} := \begin{pmatrix} W \\ 0 \end{pmatrix} \quad \text{and } \widetilde{h} = \begin{pmatrix} h \\ b \end{pmatrix}$$

2-Stage Stochastic Linear Programming

$$\min_{x \in \mathbb{R}^n} \quad c^\top x + \mathbb{E}\left[\begin{array}{ll} \min\limits_{y \in \mathbb{R}^m} & \mathbf{q}^\top y \\ \text{s.t.} & Tx + Wy \leqslant h \\ & Ax + 0y \leqslant b \end{array}\right]$$

where $T \in \mathbb{R}^{p \times n}, W \in \mathbb{R}^{p \times m}$ and $h \in \mathbb{R}^p$.
We can assume $A = 0$ and $b = 0$:
We set

$$\widetilde{T} := \begin{pmatrix} T \\ A \end{pmatrix}, \quad \widetilde{W} := \begin{pmatrix} W \\ 0 \end{pmatrix} \quad \text{and } \widetilde{h} = \begin{pmatrix} h \\ b \end{pmatrix}$$

2-Stage Stochastic Linear Programming

$$\min_{x \in \mathbb{R}^n} \quad c^\top x + \mathbb{E}\left[\begin{array}{ll} \min\limits_{y \in \mathbb{R}^m} & \mathbf{q}^\top y \\ \text{s.t.} & \widetilde{T}x + \widetilde{W}y \leqslant \widetilde{h} \\ & \end{array}\right]$$

where $T \in \mathbb{R}^{p \times n}, W \in \mathbb{R}^{p \times m}$ and $h \in \mathbb{R}^p$.
We can assume $A = 0$ and $b = 0$:
We set

$$\widetilde{T} := \begin{pmatrix} T \\ A \end{pmatrix}, \quad \widetilde{W} := \begin{pmatrix} W \\ 0 \end{pmatrix} \quad \text{and } \widetilde{h} = \begin{pmatrix} h \\ b \end{pmatrix}$$

2-Stage Stochastic Linear Programming

$$\min_{x \in \mathbb{R}^n} c^\top x + V(x) \tag{2SLP}$$

where

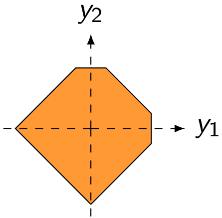
$$V(x) := \mathbb{E}\left[\begin{array}{ll} \min\limits_{y \in \mathbb{R}^m} & \mathbf{q}^\top y \\ \text{s.t.} & Tx + Wy \leqslant h \end{array}\right]$$

Fiber P_x

$$V(x) = \mathbb{E}\Big[\min_{y \in P_x} \mathbf{q}^\top y\Big] \quad \text{where} \quad P_x := \{y \in \mathbb{R}^m \mid Tx + Wy \leqslant h\}$$

We assume $\text{supp}(\mathbf{q}) \subset -\text{Cone}(W^\top)$ i.e. $V(x) > -\infty$. Example:

$$T = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ -1 \\ 1 \end{pmatrix} \quad W = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -1 & -1 \\ -1 & 1 \\ 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{pmatrix} \quad h = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$



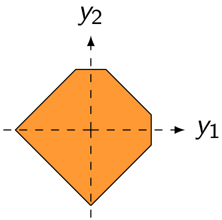
P_x for $x = 0.8$

Fiber P_x

$$V(x) = \mathbb{E}\Big[\min_{y \in P_x} \mathbf{q}^\top y\Big] \quad \text{where} \quad P_x := \{y \in \mathbb{R}^m \mid Tx + Wy \leqslant h\}$$

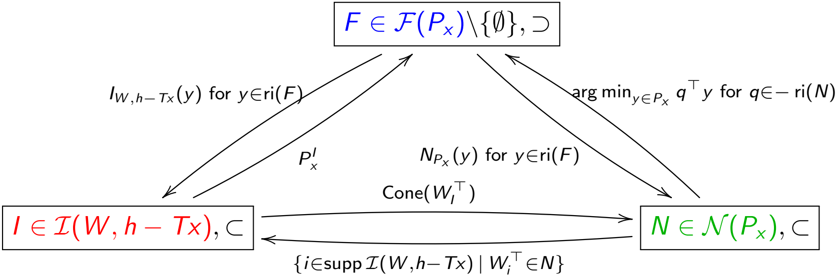
We assume $\mathrm{supp}(\mathbf{q}) \subset -\mathrm{Cone}(W^\top)$ i.e. $V(x) > -\infty$. Example:

$$
\begin{aligned}
y_1 + y_2 &\leqslant 1 && (1)\\
y_1 - y_2 &\leqslant 1 && (2)\\
-y_1 - y_2 &\leqslant 1 && (3)\\
-y_1 + y_2 &\leqslant 1 && (4)\\
y_1 &\leqslant x && (5)\\
y_2 &\leqslant x && (6)\\
x &\leqslant 1.5 && (7)
\end{aligned}
$$



P_x for $x = 0.8$

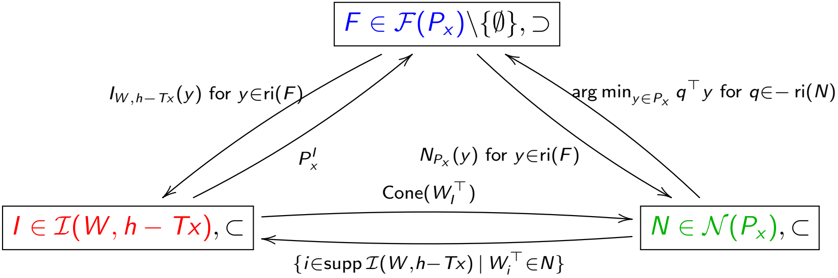
Expectation to final sum



$$V(x) = \mathbb{E}\Big[\min_{y \in P_x} \mathbf{q}^\top y\Big]$$

$$= \sum_{N \in \mathcal{N}(P_x)} \mathbb{E}\big[\mathbf{q}^\top \mathbb{1}_{\mathbf{q} \in -\mathrm{ri}\, N}\big] y_N(x) \quad \text{with } y_N(x) \in \cap_{q \in -N} \arg\min_{y \in P_x} q^\top y$$

Expectation to final sum



$$V(x) = \mathbb{E}\big[\min_{y \in P_x} \mathbf{q}^\top y\big]$$

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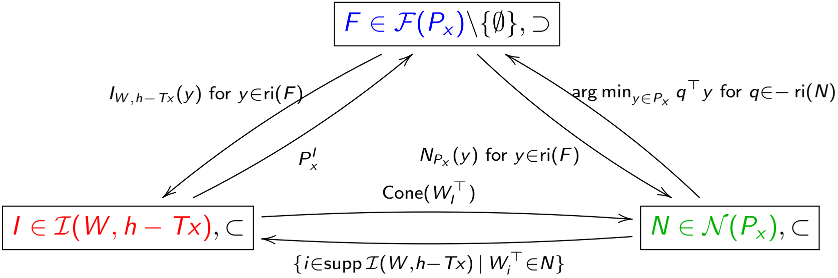
$$= \sum_{F \in \mathcal{F}(P_x)} \mathbb{E}\big[\mathbf{q}^\top \mathbb{1}_{\mathbf{q} \in -\mathrm{ri}\, N_{P_x}(F)}\big] y_F \quad \text{with } y_F \in F$$

Expectation to final sum



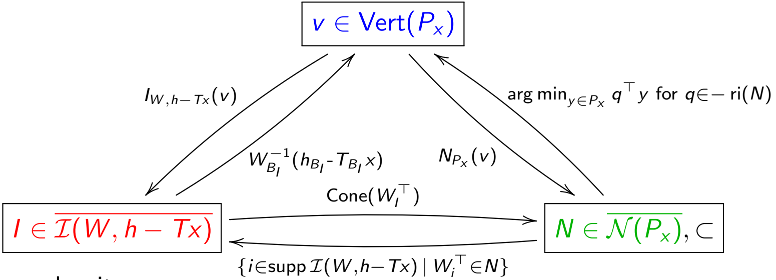
$$V(x) = \mathbb{E}\big[\min_{y\in P_x} \mathbf{q}^\top y\big]$$

$$= \sum_{N\in\mathcal{N}(P_x)} \mathbb{E}\big[\mathbf{q}^\top \mathbb{1}_{\mathbf{q}\in -\mathrm{ri}\,N}\big] y_N(x) \quad \text{with } y_N(x) \in \cap_{q\in -N} \arg\min_{y\in P_x} q^\top y$$

$$= \sum_{F\in\mathcal{F}(P_x)} \mathbb{E}\big[\mathbf{q}^\top \mathbb{1}_{\mathbf{q}\in -\mathrm{ri}\,N_{P_x}(F)}\big] y_F \quad \text{with } y_F \in F$$

$$= \sum_{I\in\mathcal{I}(W,h-Tx)} \mathbb{E}\big[\mathbf{q}^\top \mathbb{1}_{\mathbf{q}\in -\mathrm{ri}\,\mathrm{Cone}(W_I^\top)}\big] y_I(x) \quad \text{with } y_I(x) \in P_x^I$$

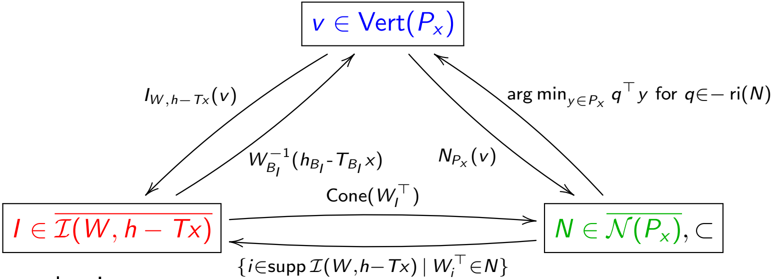
Expectation to final sum



If $\mathbf{q}$ has a density,

$$
\begin{aligned}
V(x) &= \mathbb{E}\big[\min_{y\in P_x} \mathbf{q}^\top y\big] \\
&= \sum_{N\in\overline{\mathcal{N}(P_x)}} \mathbb{E}\big[\mathbf{q}^\top \mathbb{1}_{\mathbf{q}\in -N}\big] y_N(x) \quad \text{with } y_N(x) \in \cap_{q\in -N} \arg\min_{y\in P_x} q^\top y \\
&= \sum_{v\in\mathrm{Vert}(P_x)} \mathbb{E}\big[\mathbf{q}^\top \mathbb{1}_{\mathbf{q}\in -N_{P_x}(F)}\big] v \\
&= \sum_{I\in\overline{\mathcal{I}(W,h-Tx)}} \mathbb{E}\big[\mathbf{q}^\top \mathbb{1}_{\mathbf{q}\in -\mathrm{Cone}(W_I^\top)}\big] y_I(x) \quad \text{with } y_I(x) \in P_x^I
\end{aligned}
$$

Expectation to final sum



If $\mathbf{q}$ has a density,

$$V(x) = \mathbb{E}\big[\min_{y \in P_x} \mathbf{q}^\top y\big]$$

$$= \sum_{N \in \overline{\mathcal{N}(P_x)}} \mathbb{E}\big[\mathbf{q}^\top \mathbb{1}_{\mathbf{q} \in -N}\big] y_N(x) \quad \text{with } y_N(x) \in \cap_{q \in -N} \arg\min_{y \in P_x} q^\top y$$

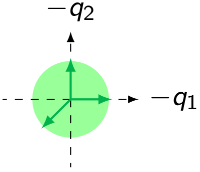
$$= \sum_{v \in \mathrm{Vert}(P_x)} \mathbb{E}\big[\mathbf{q}^\top \mathbb{1}_{\mathbf{q} \in -N_{P_x}(F)}\big] v$$

$$= \sum_{I \in \overline{\mathcal{I}(W,h-Tx)}} \mathbb{E}\big[\mathbf{q}^\top \mathbb{1}_{\mathbf{q} \in -\mathrm{Cone}(W_I^\top)}\big] W_{B_I}^{-1}(h_{B_I} - T_{B_I} x) \text{ with basis } B_I \subset I$$

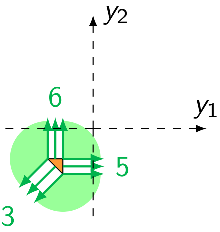
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

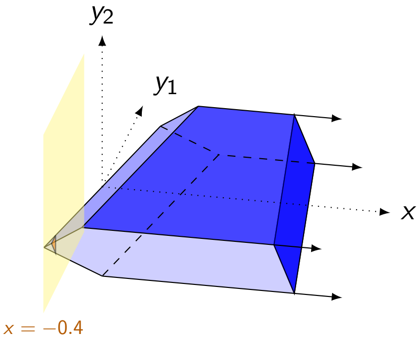
For $x = -0.4$, $\quad \overline{\mathcal{I}(W, h - Tx)} = \{53, 36, 65\}$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

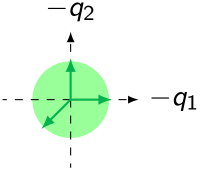


P and P_x

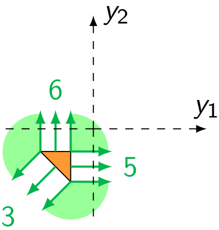
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

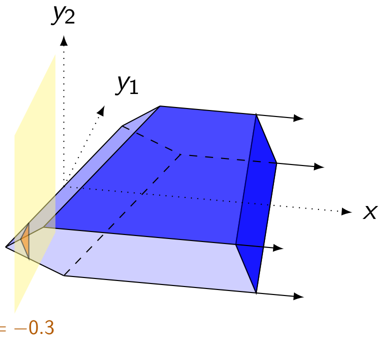
For $x = -0.3$, $\quad \overline{\mathcal{I}(W, h - Tx)} = \{53, 36, 65\}$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

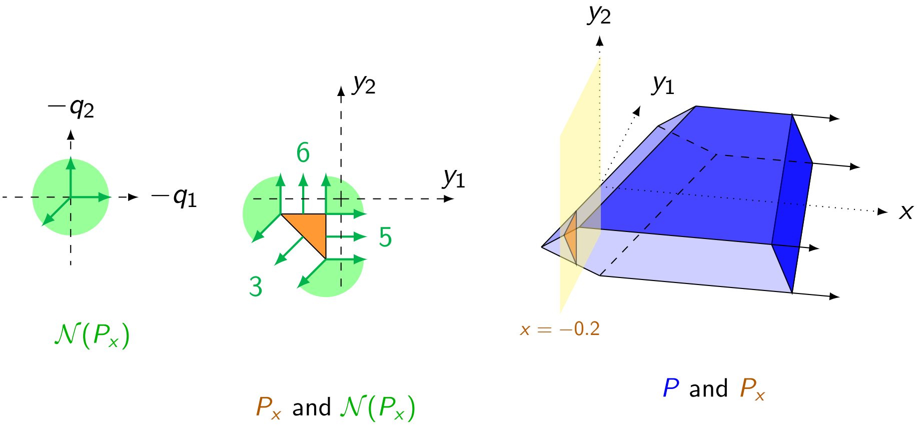


P and P_x

$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

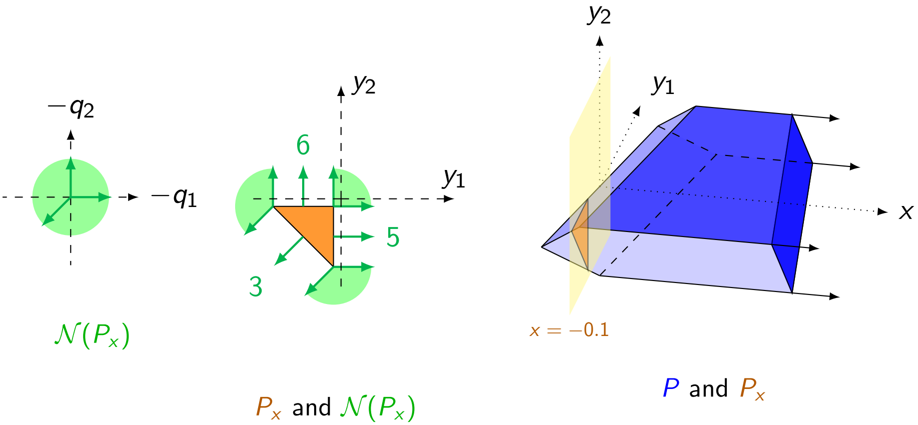
$$\text{For } x = -0.2, \quad \overline{\mathcal{I}(W, h - Tx)} = \{53, 36, 65\}$$



$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

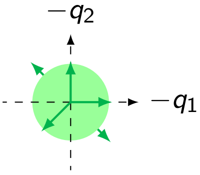
For $x = -0.1$, $\overline{\mathcal{I}(W, h - Tx)} = \{53, 36, 65\}$



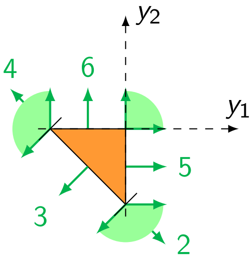
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

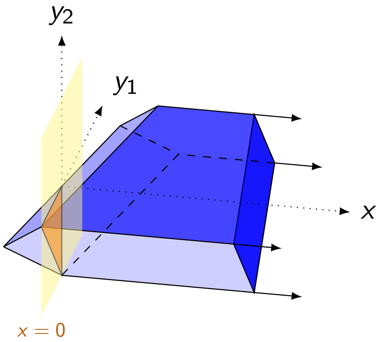
For $x = 0$, $\overline{\mathcal{I}(W, h - Tx)} = \{523, 346, 65\}$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

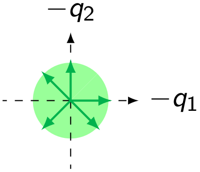


P and P_x

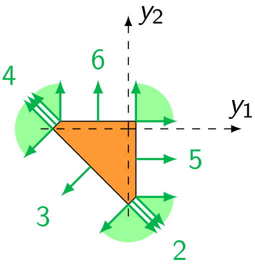
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

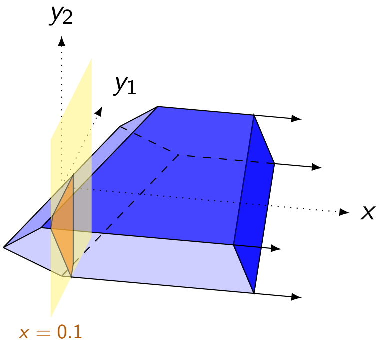
$$\text{For } x = 0.1, \quad \overline{\mathcal{I}(W, h - Tx)} = \{52, 23, 34, 46, 65\}$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

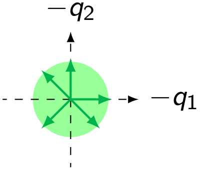


P and P_x

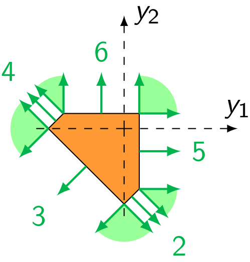
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

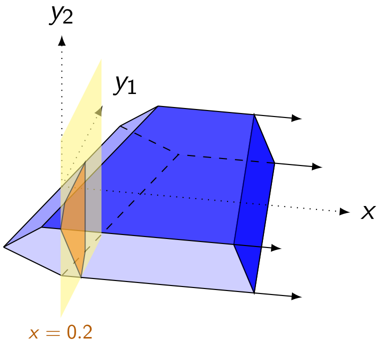
For $x = 0.2$, $\quad \overline{\mathcal{I}(W, h - Tx)} = \{52, 23, 34, 46, 65\}$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

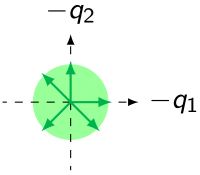


P and P_x

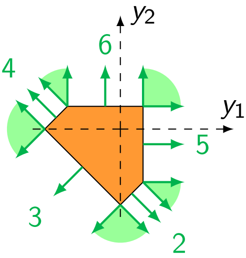
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

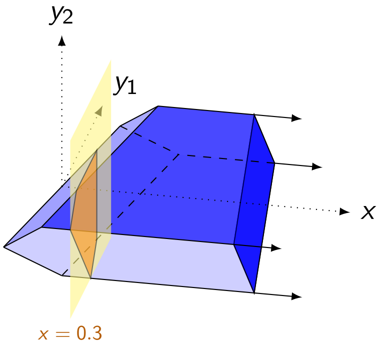
For $x = 0.3$, $\quad \overline{\mathcal{I}(W, h - Tx)} = \{52, 23, 34, 46, 65\}$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

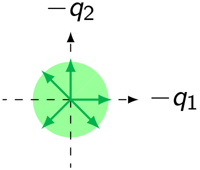


P and P_x

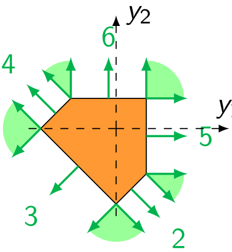
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

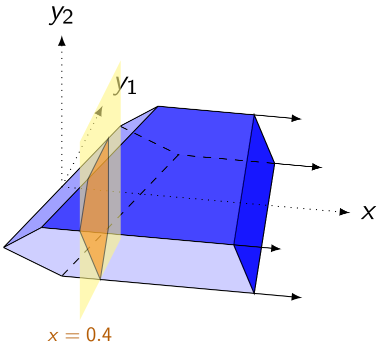
$$\text{For } x = 0.4, \quad \overline{\mathcal{I}(W, h - Tx)} = \{52, 23, 34, 46, 65\}$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

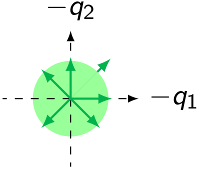


P and P_x

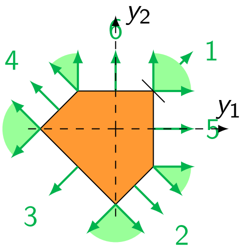
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

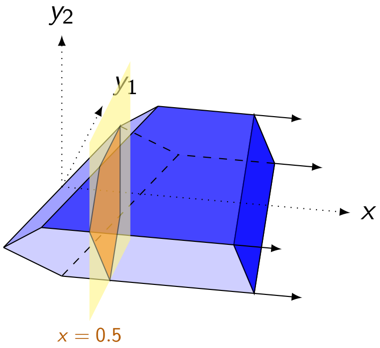
For $x = 0.5$, $\quad \overline{\mathcal{I}(W, h - Tx)} = \{52, 23, 34, 46, 615\}$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

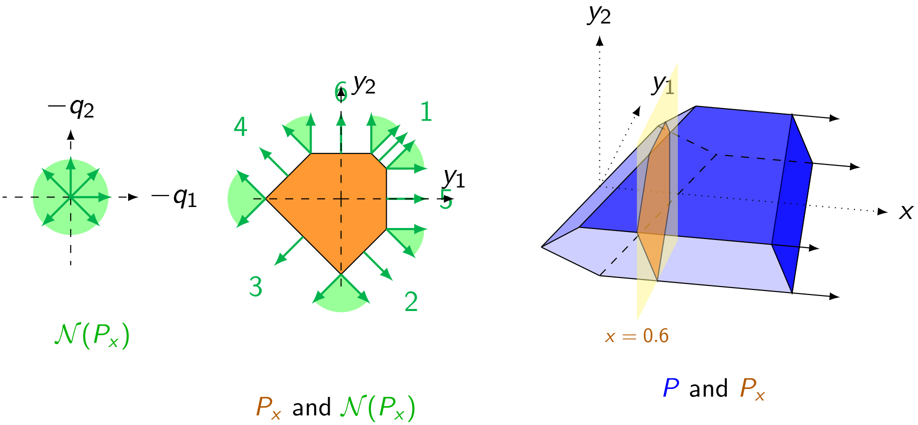


P and P_x

$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

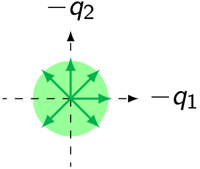
For $x = 0.6$, $\overline{\mathcal{I}(W, h - Tx)} = \{52, 23, 34, 46, 61, 15\}$



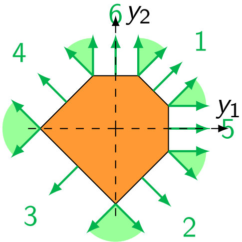
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

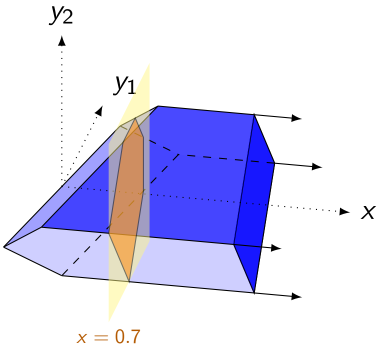
$$\text{For } x = 0.7, \quad \overline{\mathcal{I}(W, h - Tx)} = \{52, 23, 34, 46, 61, 15\}$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

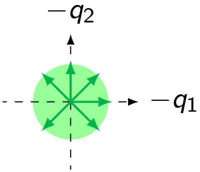


P and P_x

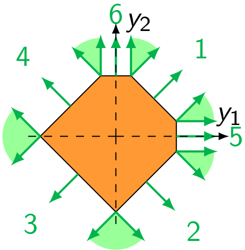
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

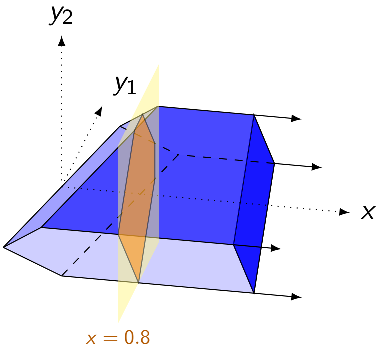
$$\text{For } x = 0.8, \quad \overline{\mathcal{I}(W, h - Tx)} = \{52, 23, 34, 46, 61, 15\}$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

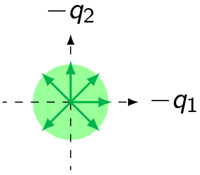


P and P_x

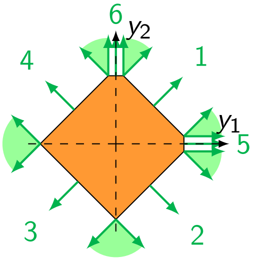
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

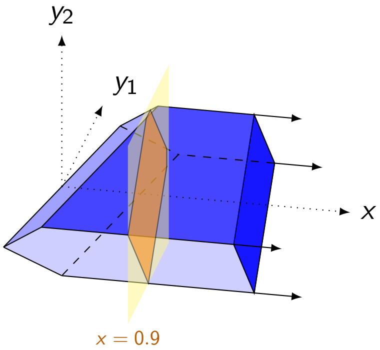
$$\text{For } x = 0.9, \quad \overline{\mathcal{I}(W, h - Tx)} = \{52, 23, 34, 46, 61, 15\}$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

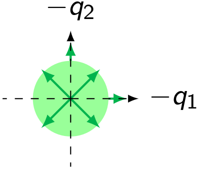


P and P_x

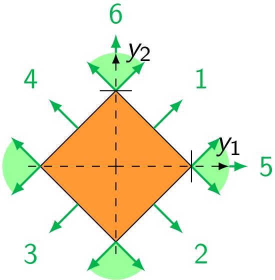
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

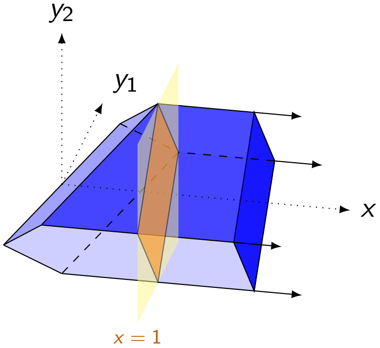
$$\text{For } x = 1, \quad \overline{\mathcal{I}(W, h - Tx)} = \{152, 23, 34, 461\}$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

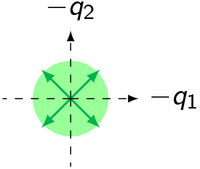


P and P_x

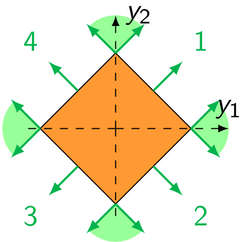
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

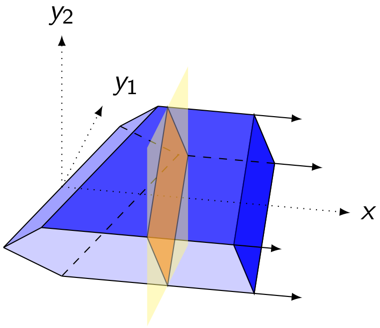
$$\text{For } x = 1.1, \quad \overline{\mathcal{I}(W, h - Tx)} = \{12, 23, 34, 41\}$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

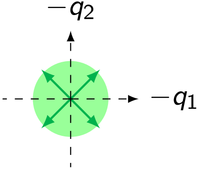


P and P_x

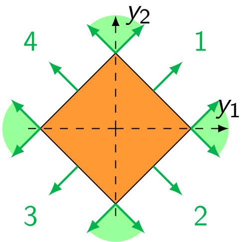
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

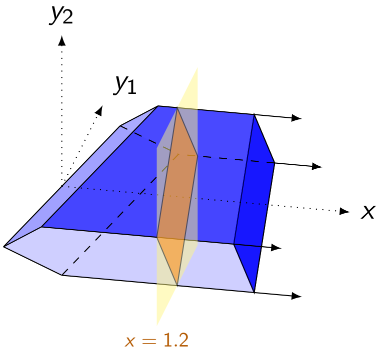
$$\text{For } x = 1.2, \quad \overline{\mathcal{I}(W, h - Tx)} = \{12, 23, 34, 41\}$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

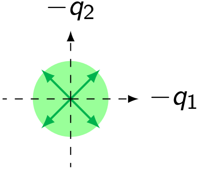


P and P_x

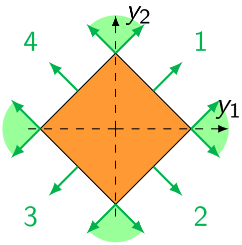
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

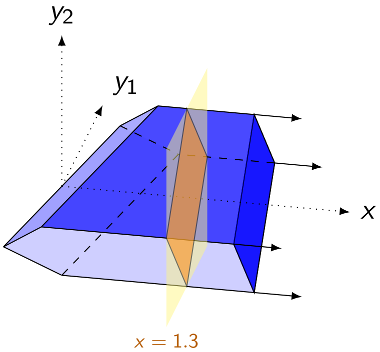
$$\text{For } x = 1.3, \quad \overline{\mathcal{I}(W, h - Tx)} = \{12, 23, 34, 41\}$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

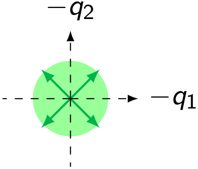


P and P_x

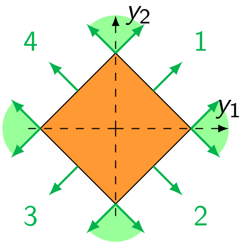
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

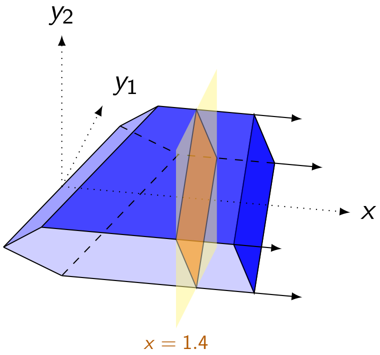
$$\text{For } x = 1.4, \quad \overline{\mathcal{I}(W, h - Tx)} = \{12, 23, 34, 41\}$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$



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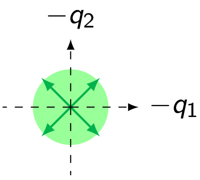
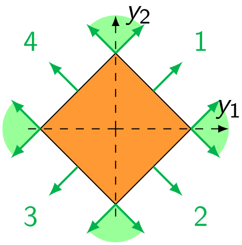
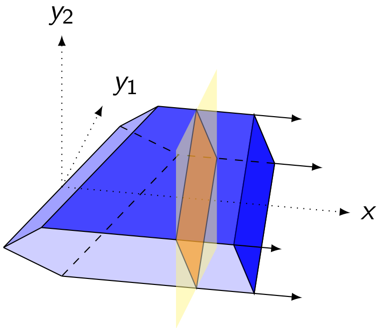
$$\text{For } x = 1.4, \quad \overline{\mathcal{I}(W, h - Tx)} = \{12, 23, 34, 41\}$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

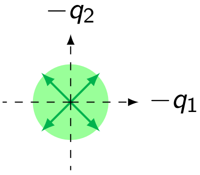


P and P_x

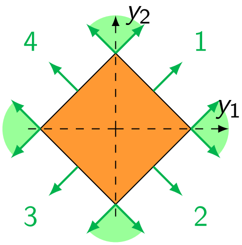
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

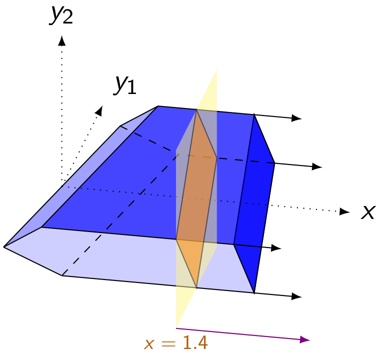
For $x = 1.4$, $\quad \overline{\mathcal{I}(W, h - Tx)} = \{12, 23, 34, 41\}$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

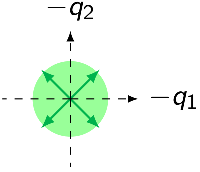


P and P_x

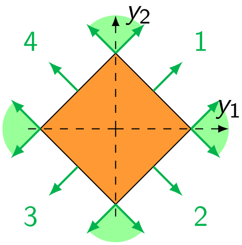
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

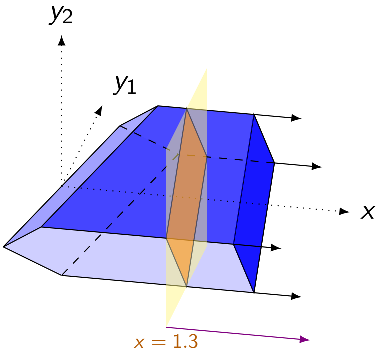
For $x = 1.3$, $\quad \overline{\mathcal{I}(W, h - Tx)} = \{12, 23, 34, 41\}$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

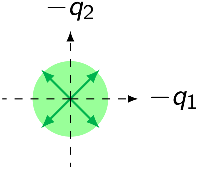


P and P_x

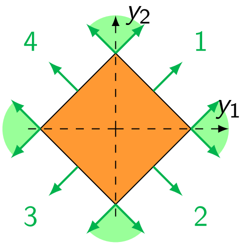
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

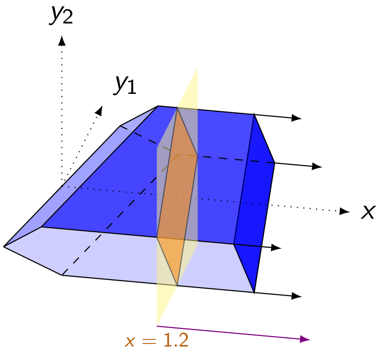
For $x = 1.2$, $\quad \overline{\mathcal{I}(W, h - Tx)} = \{12, 23, 34, 41\}$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

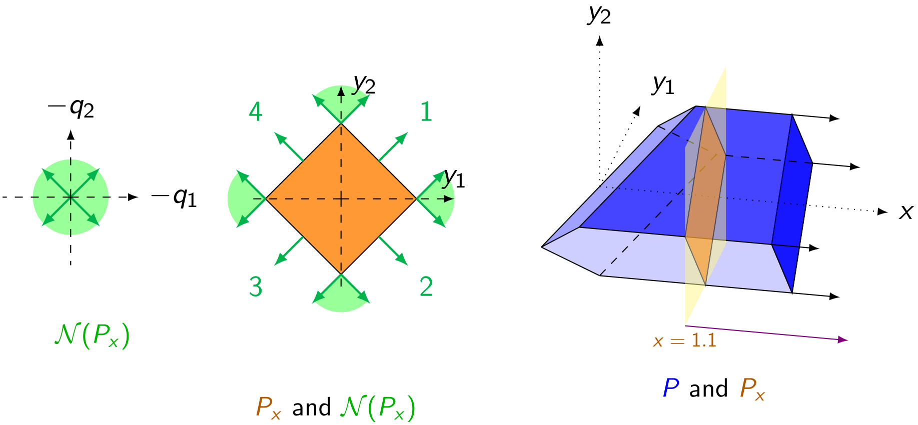


P and P_x

$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

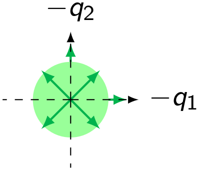
For $x = 1.1$, $\quad \overline{\mathcal{I}(W, h - Tx)} = \{12, 23, 34, 41\}$



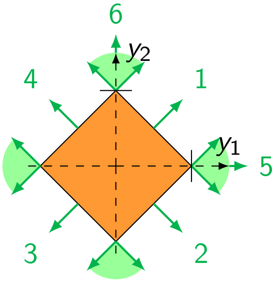
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

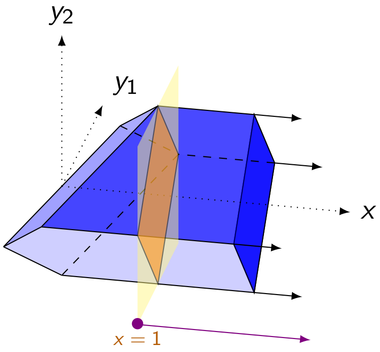
For $x = 1$, $\quad \overline{\mathcal{I}(W, h - Tx)} = \{152, 23, 34, 461\}$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

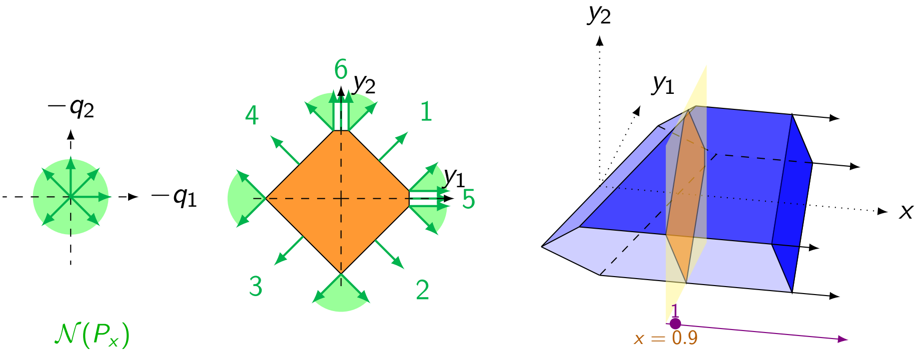


P and P_x

$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

$$\text{For } x = 0.9, \quad \overline{\mathcal{I}(W, h - Tx)} = \{52, 23, 34, 46, 61, 15\}$$



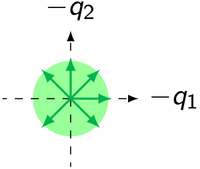
P_x and $\mathcal{N}(P_x)$

P and P_x

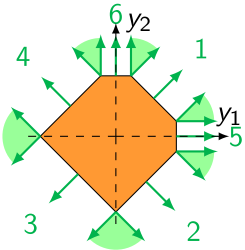
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

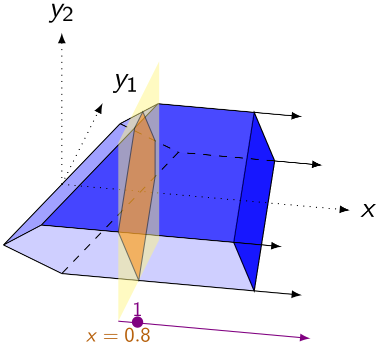
$$\text{For } x = 0.8, \quad \overline{\mathcal{I}(W, h - Tx)} = \{52, 23, 34, 46, 61, 15\}$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

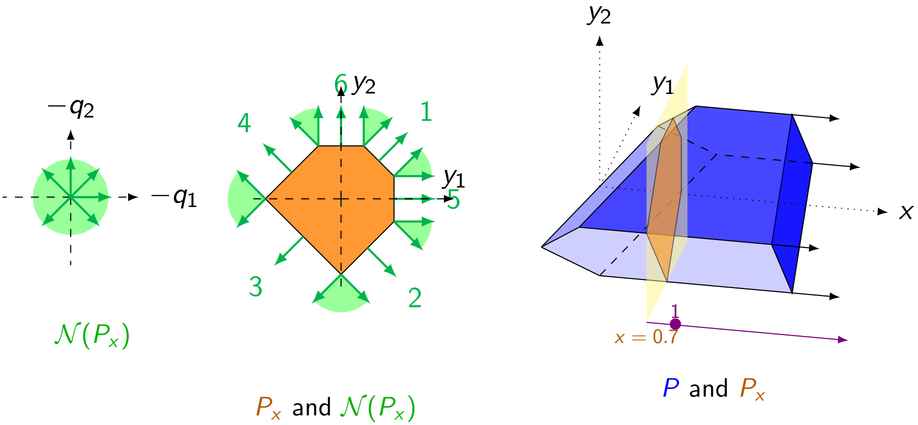


P and P_x

$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

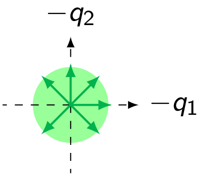
$$\text{For } x = 0.7, \quad \overline{\mathcal{I}(W, h - Tx)} = \{52, 23, 34, 46, 61, 15\}$$



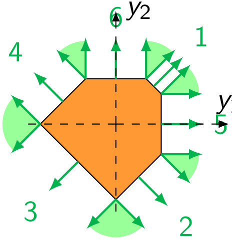
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

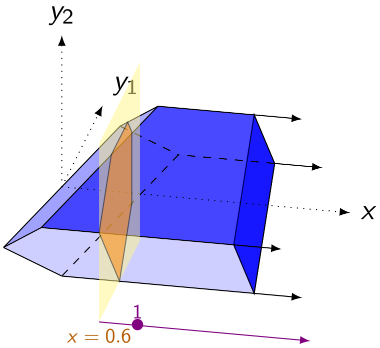
$$\text{For } x = 0.6, \quad \overline{\mathcal{I}(W, h - Tx)} = \{52, 23, 34, 46, 61, 15\}$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

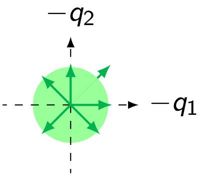


P and P_x

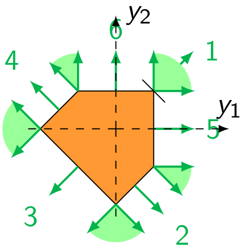
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

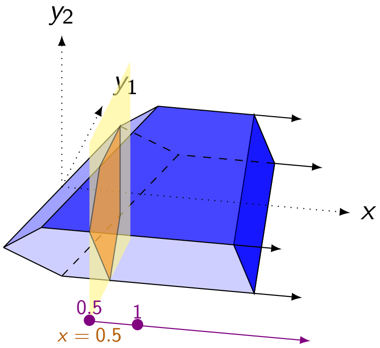
$$\text{For } x = 0.5, \quad \overline{\mathcal{I}(W, h - Tx)} = \{52, 23, 34, 46, 615\}$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

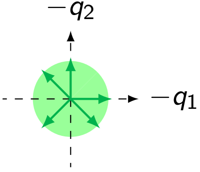


P and P_x

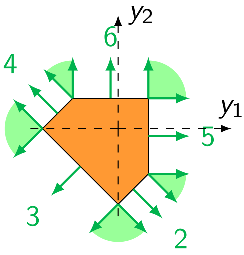
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

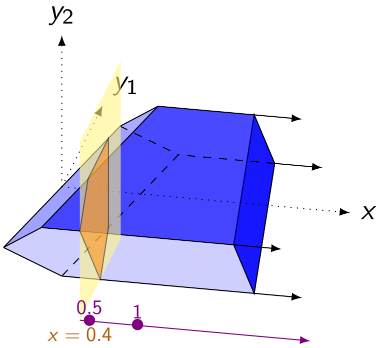
For $x = 0.4$, $\quad \overline{\mathcal{I}(W, h - Tx)} = \{52, 23, 34, 46, 65\}$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

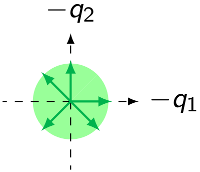


P and P_x

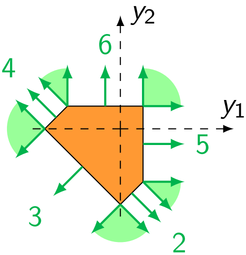
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

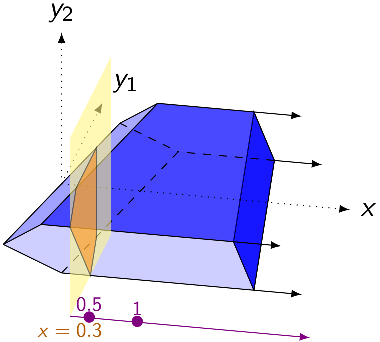
For $x = 0.3$, $\quad \overline{\mathcal{I}(W, h - Tx)} = \{52, 23, 34, 46, 65\}$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

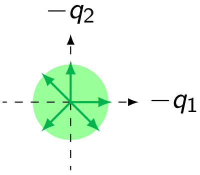


P and P_x

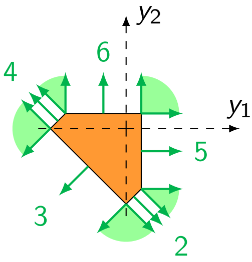
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

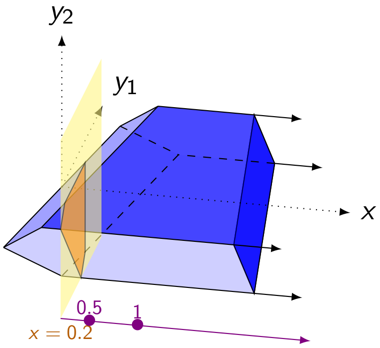
$$\text{For } x = 0.2, \quad \overline{\mathcal{I}(W, h - Tx)} = \{52, 23, 34, 46, 65\}$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

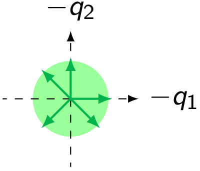


P and P_x

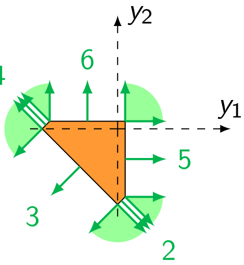
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

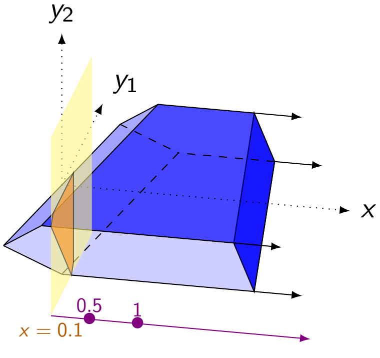
$$\text{For } x = 0.1, \quad \overline{\mathcal{I}(W, h - Tx)} = \{52, 23, 34, 46, 65\}$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

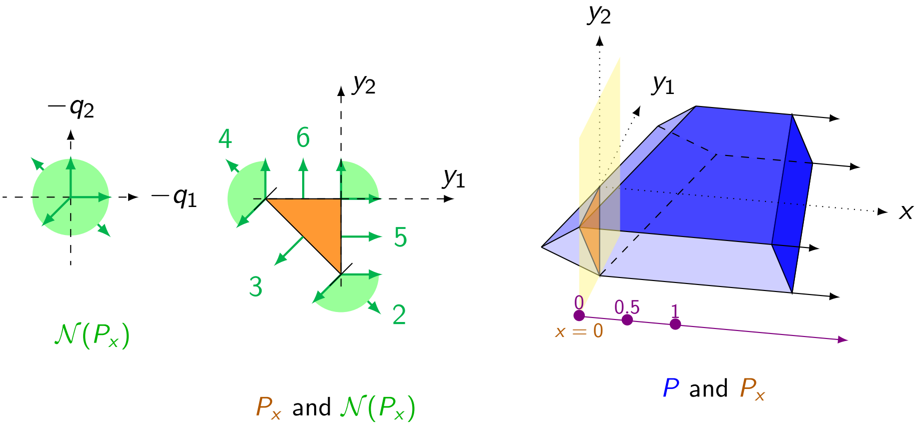


P and P_x

$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

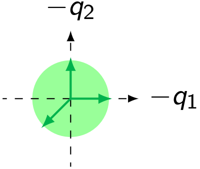
$$\text{For } x = 0, \quad \overline{\mathcal{I}(W, h - Tx)} = \{523, 346, 65\}$$



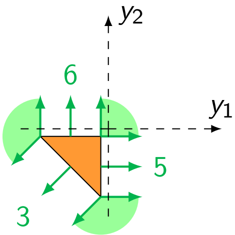
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

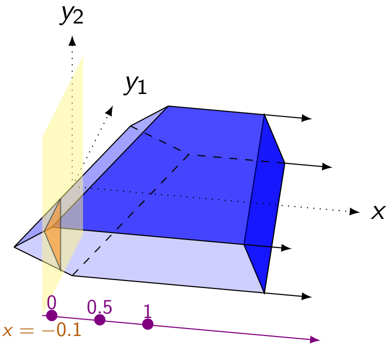
$$\text{For } x = -0.1, \quad \overline{\mathcal{I}(W, h - Tx)} = \{53, 36, 65\}$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

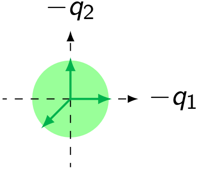


P and P_x

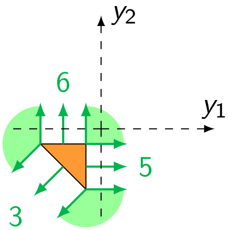
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

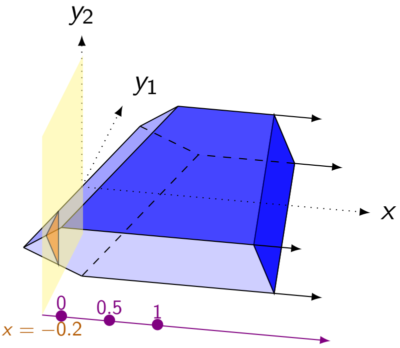
$$\text{For } x = -0.2, \quad \overline{\mathcal{I}(W, h - Tx)} = \{53, 36, 65\}$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

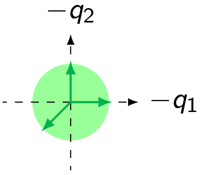


P and P_x

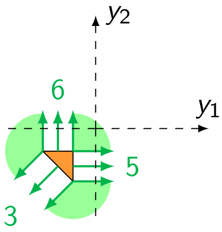
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

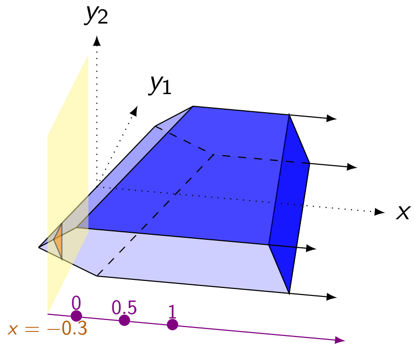
For $x = -0.3$, $\quad \overline{\mathcal{I}(W, h - Tx)} = \{53, 36, 65\}$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

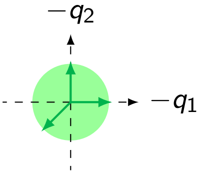


P and P_x

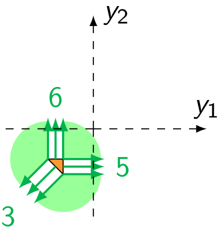
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

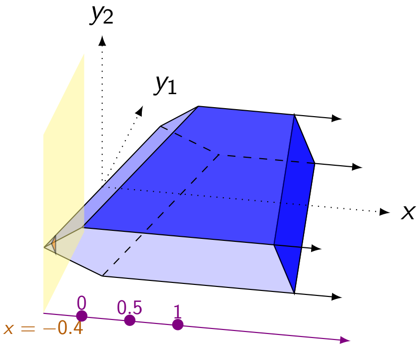
$$\text{For } x = -0.4, \quad \overline{\mathcal{I}(W, h - Tx)} = \{53, 36, 65\}$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

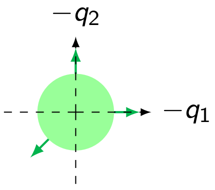


P and P_x

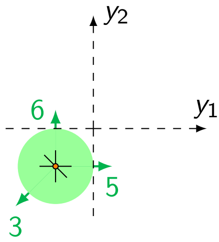
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

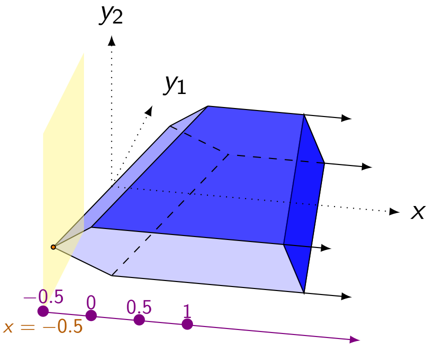
$$\text{For } x = -0.5, \quad \overline{\mathcal{I}(W, h - Tx)} = \{536\}$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$



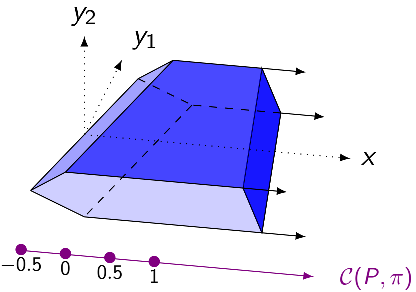
P and P_x

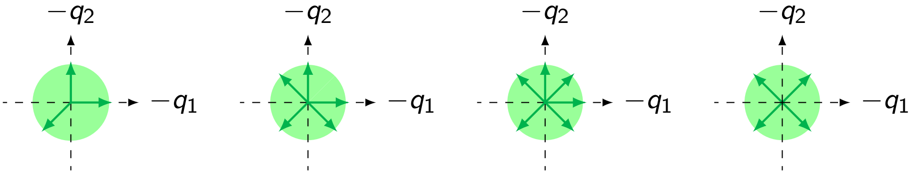
What are the constant regions of $\mathcal{N}(P_x)$, $\mathcal{I}(W, h - Tx)$?

Lemma

There exists a collection $\mathcal{C}(P,\pi)$ whose relative interior of cells are the constant regions of $x \to \mathcal{N}(P_x)$ and $x \to \mathcal{I}(W, h - Tx)$.

For $\sigma \in \mathcal{C}(P,\pi)$ and $x, x' \in \mathrm{ri}(\sigma)$,
$\mathcal{N}(P_x) = \mathcal{N}(P_{x'}) = \mathcal{N}_\sigma$
$\mathcal{I}(W, h - Tx) = \mathcal{I}(W, h - Tx') = \mathcal{I}_\sigma$





$\mathcal{N}_\sigma$ for $\sigma = [-0.5, 0]$ $\mathcal{N}_\sigma$ for $\sigma = [0, 0.5]$ $\mathcal{N}_\sigma$ for $\sigma = [0.5, 1]$ $\mathcal{N}_\sigma$ for $\sigma = [1, +\infty)$

Chamber complex

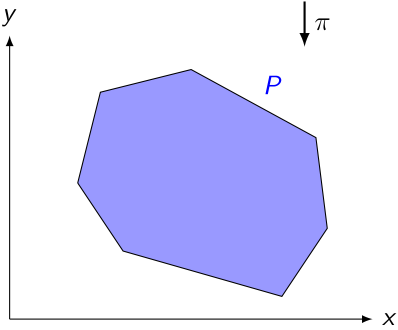
Definition

The *chamber complex* $\mathcal{C}(P, \pi)$ of P along π is

$$\mathcal{C}(P, \pi) := \{\sigma_{P,\pi}(x) \mid x \in \pi(P)\}$$

where

$$\sigma_{P,\pi}(x) := \bigcap_{F \in \mathcal{F}(P) \text{ s.t. } x \in \pi(F)} \pi(F)$$



where $\mathcal{F}(P)$ is the set of faces of P and π is the projection $(x, y) \to x$

$$\pi(E) := \{x \in \mathbb{R}^n \mid \exists y \in \mathbb{R}^m,\ (x, y) \in E\}$$

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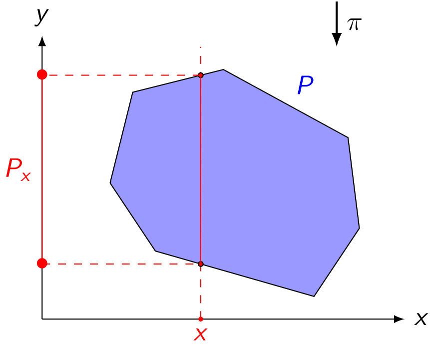

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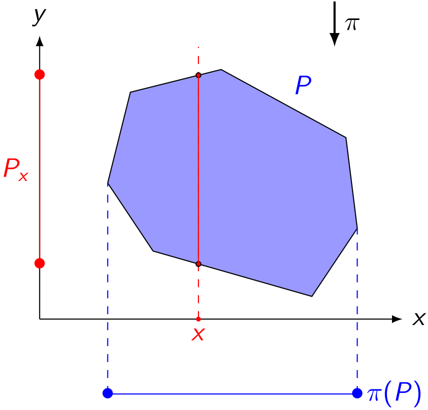

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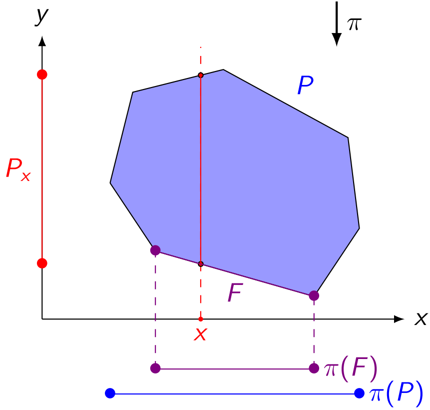

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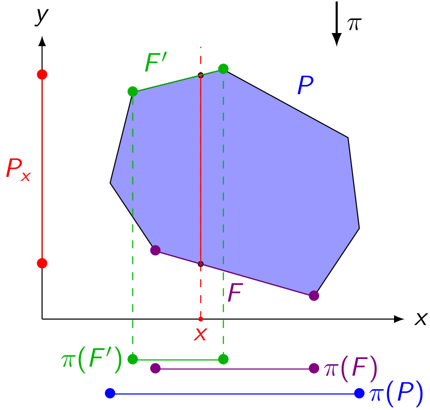

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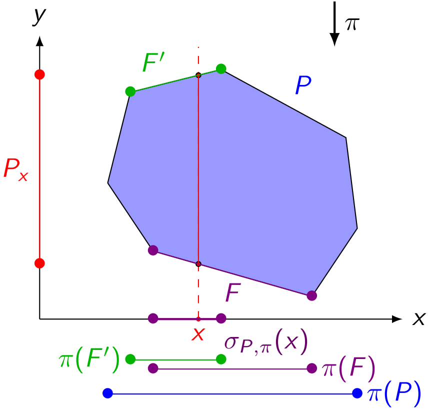

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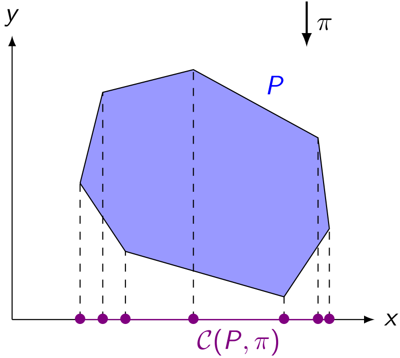

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Proof of normal equivalence

$$\mathcal{G}_x := \{G \in \mathcal{F}(P) \,|\, x \in \mathrm{ri}\,(\pi(G))\}$$

Let $\sigma \in \mathcal{C}(P, \pi)$, for all $x, x' \in \mathrm{ri}(\sigma)$, we have

$$\mathcal{G}_\sigma := \mathcal{G}_x = \mathcal{G}_{x'}$$

By the correspondences,

$$\mathcal{I}_\sigma := \mathcal{I}(W, h - Tx) = \mathcal{I}(W, h - Tx') \qquad \mathcal{N}_\sigma := \mathcal{N}(P_x) = \mathcal{N}(P_{x'})$$

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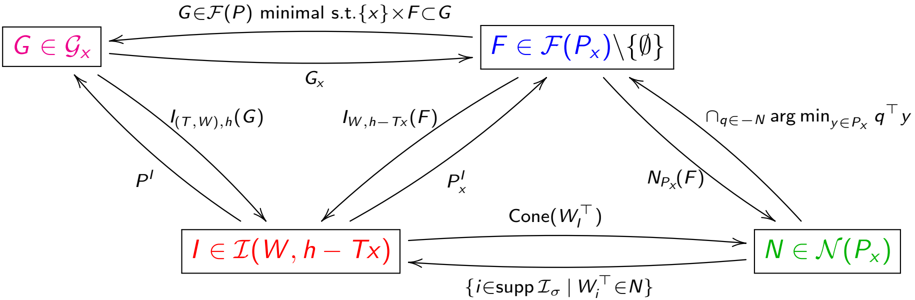

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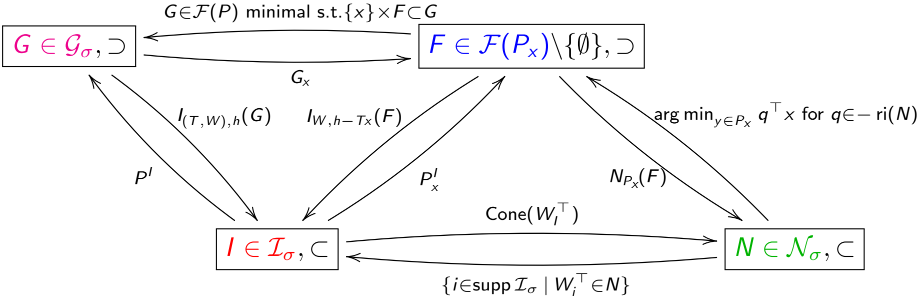
Correspondences



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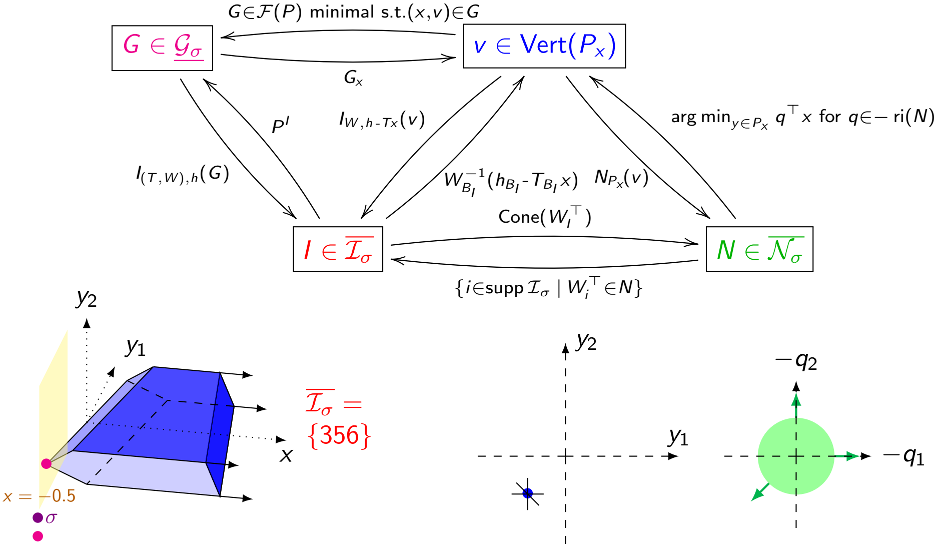

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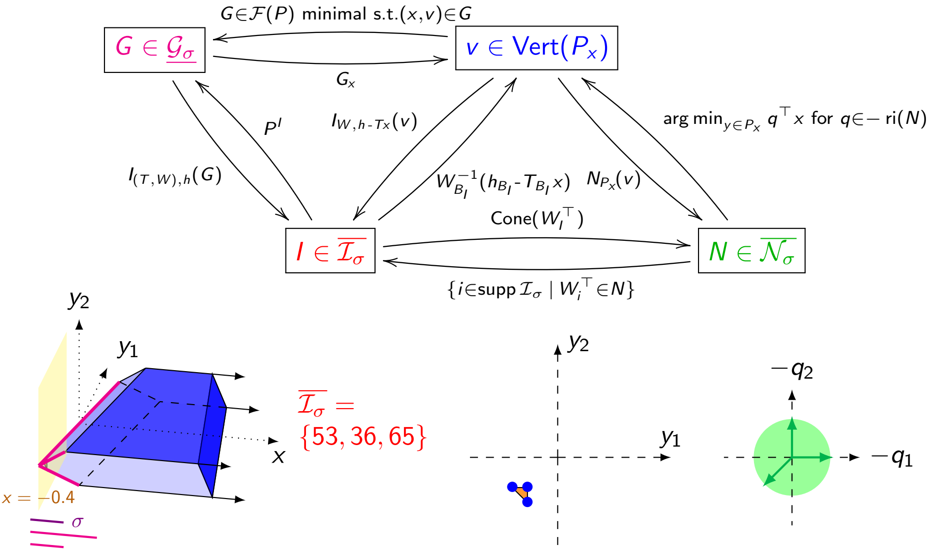

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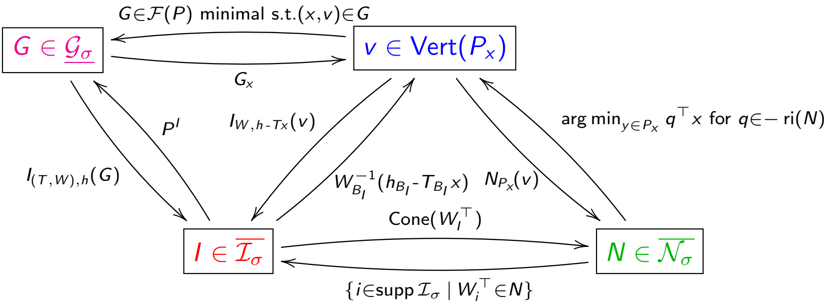
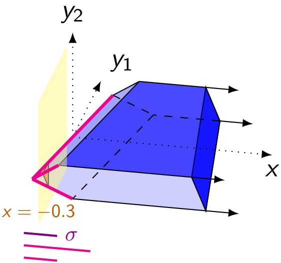
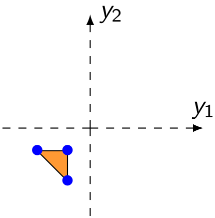
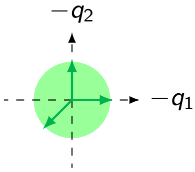




$\overline{\mathcal{I}_\sigma} = \{53, 36, 65\}$





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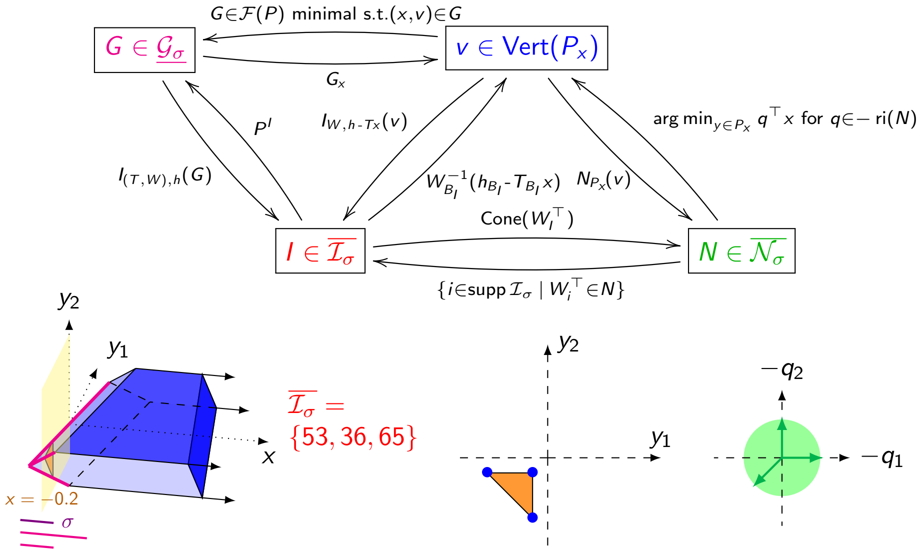

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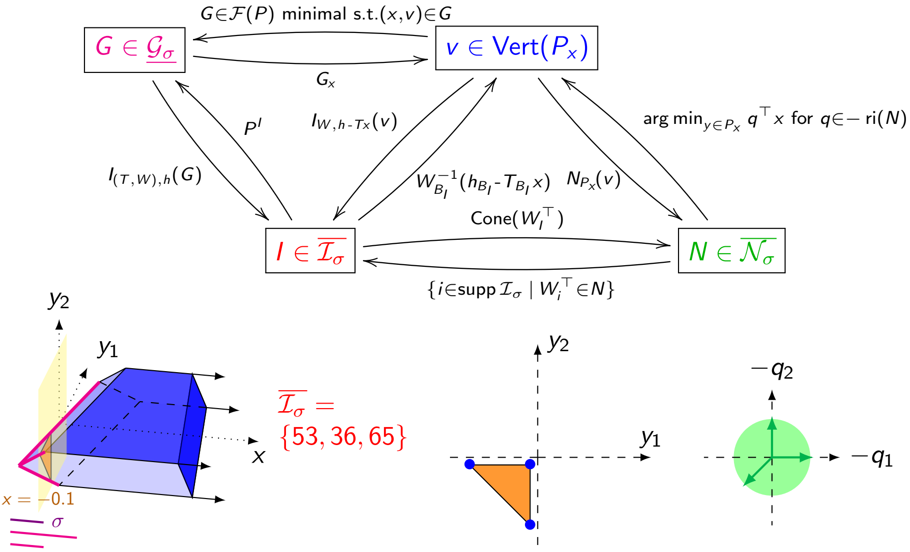

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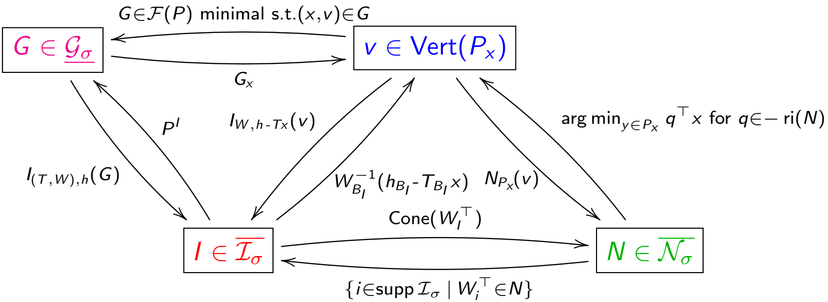

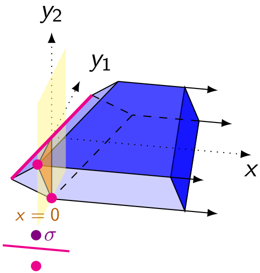

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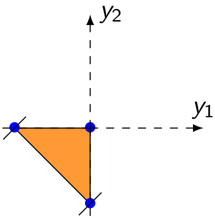

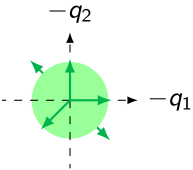

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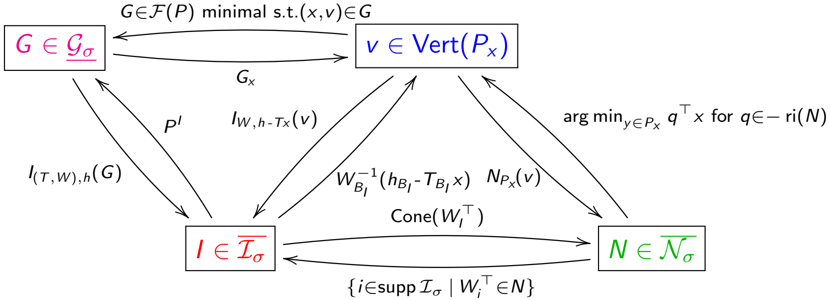

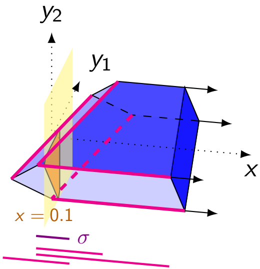

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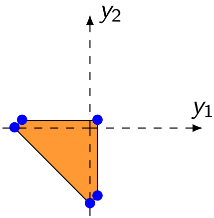

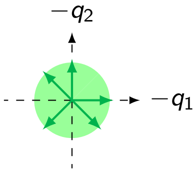

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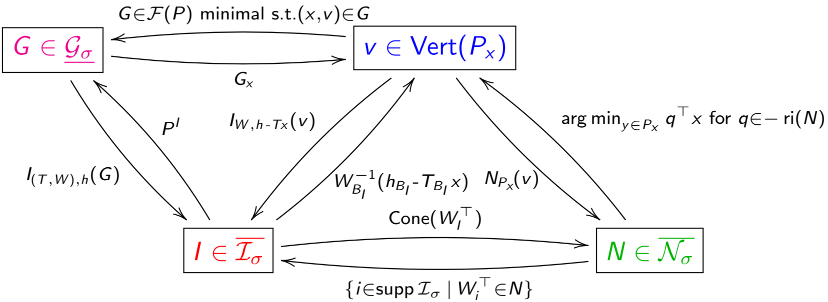
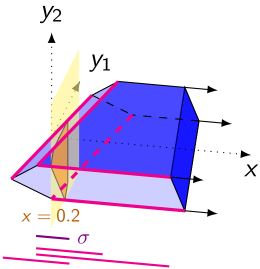
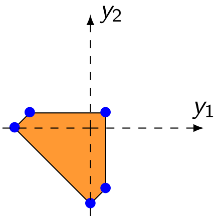
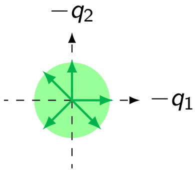




$\overline{\mathcal{I}_\sigma} = \{52, 23, 34, 46, 65\}$





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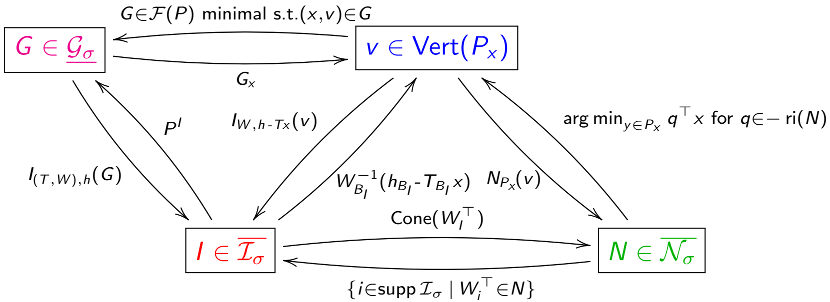
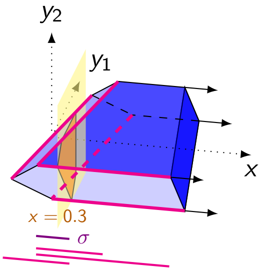
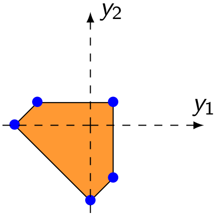
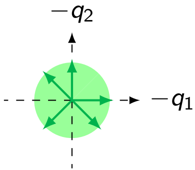




$\overline{\mathcal{I}_\sigma} = \{52, 23, 34, 46, 65\}$





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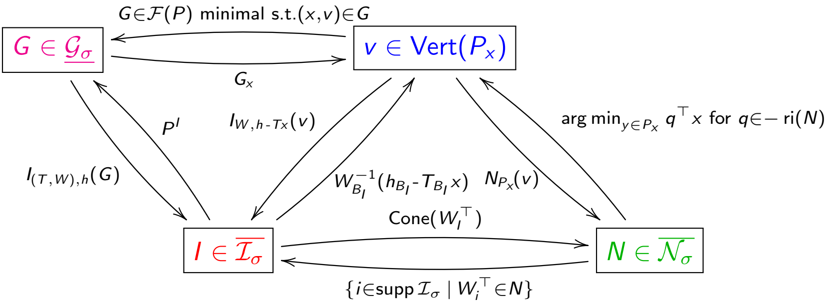
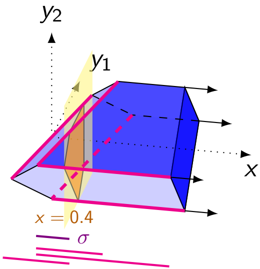
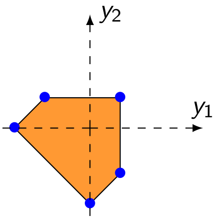
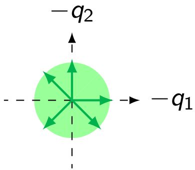




$\overline{\mathcal{I}_\sigma} = \{52, 23, 34, 46, 65\}$





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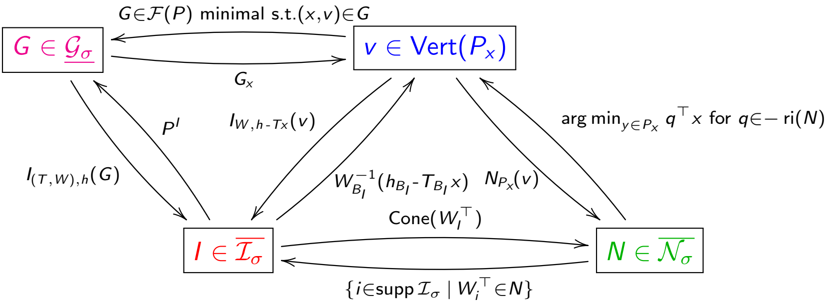
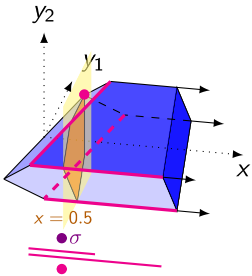
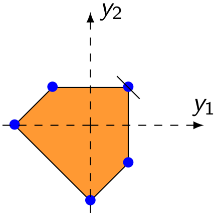
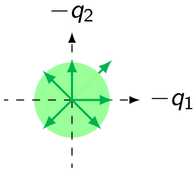




$\overline{\mathcal{I}_\sigma} = \{52, 23, 34, 46, 615\}$





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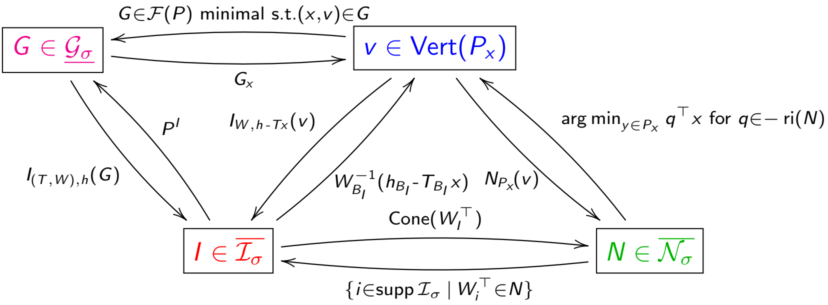
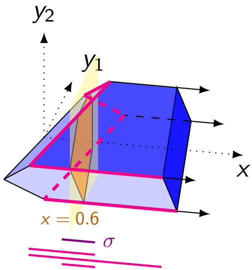
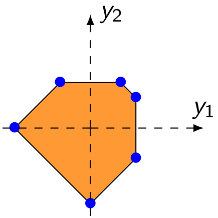
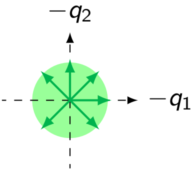




$\overline{\mathcal{I}_\sigma} =$
$\{52, 23, 34,$
$46, 61, 15\}$





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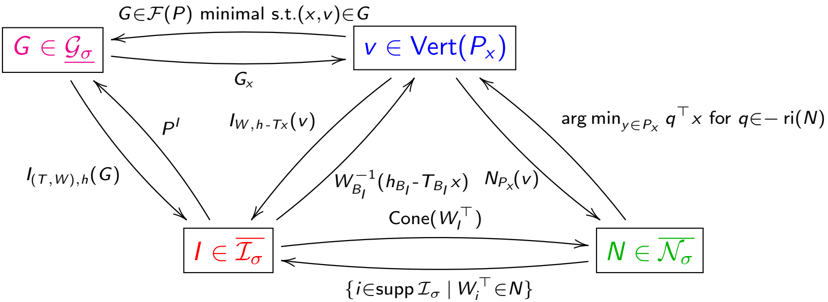
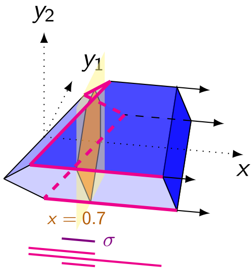
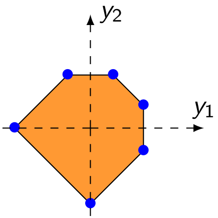
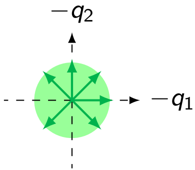




$\overline{\mathcal{I}_\sigma} = \{52, 23, 34, 46, 61, 15\}$





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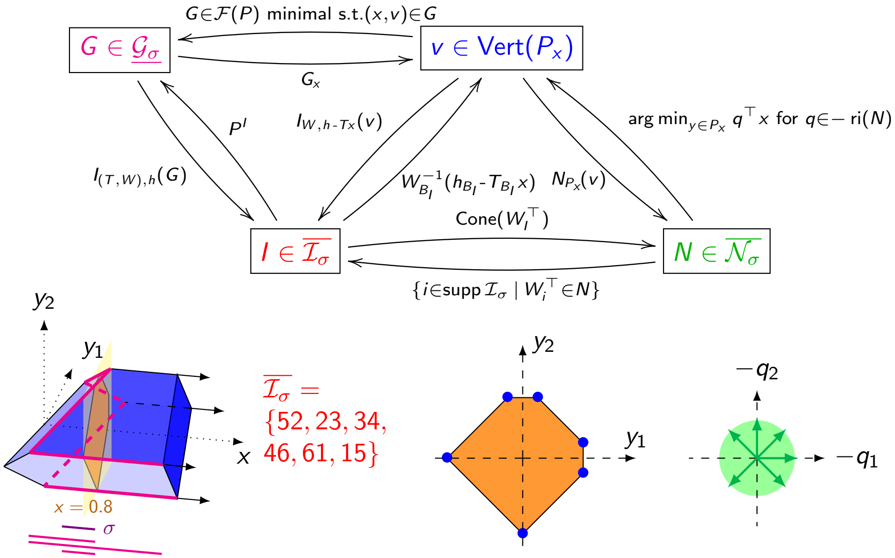

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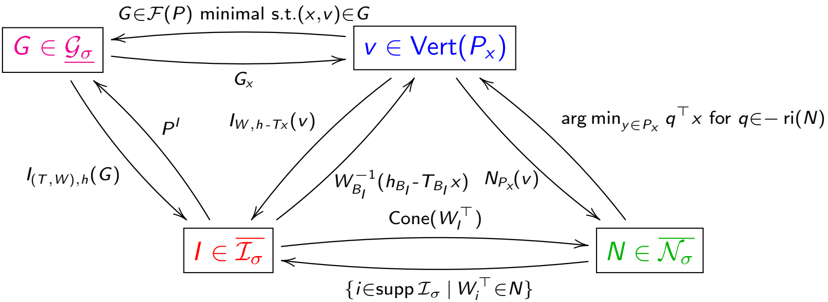
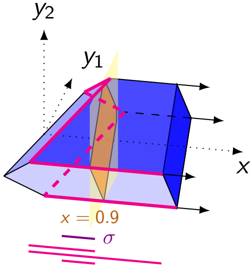
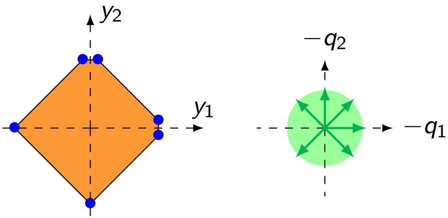




$\overline{\mathcal{I}_\sigma} = \{52, 23, 34, 46, 61, 15\}$



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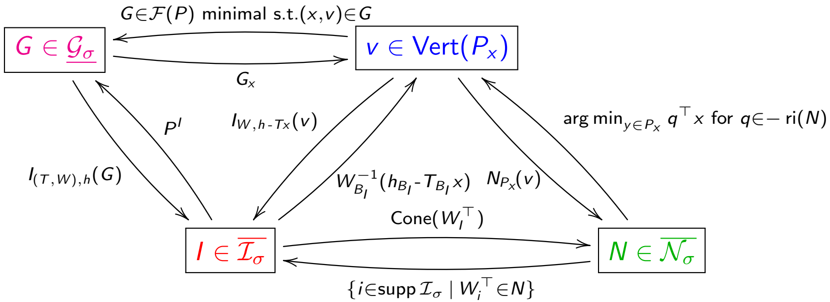
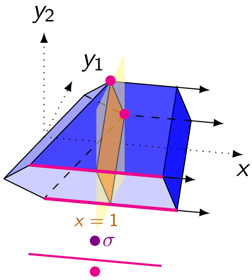
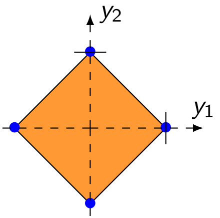
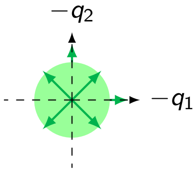




$\overline{\mathcal{I}_\sigma} = \{152, 23, 34, 461\}$





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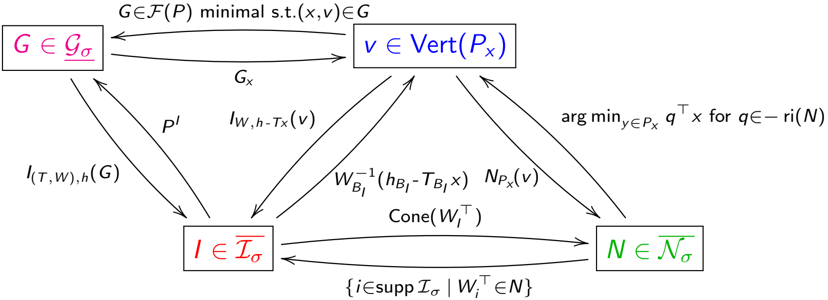
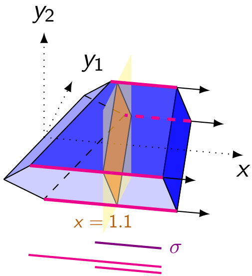
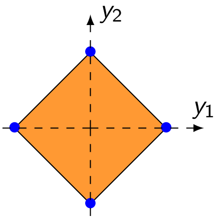
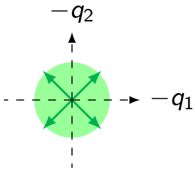




$\overline{\mathcal{I}_\sigma} =$
$\{12, 23, 34, 41\}$





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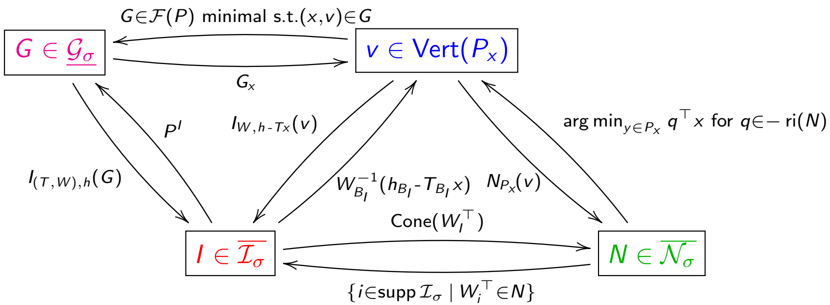
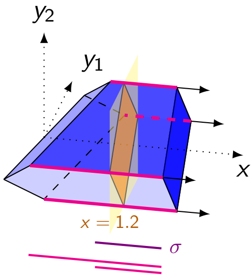
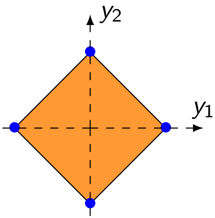
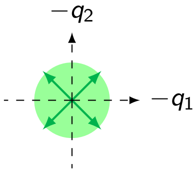




$\overline{\mathcal{I}_\sigma} =$
$\{12, 23, 34, 41\}$





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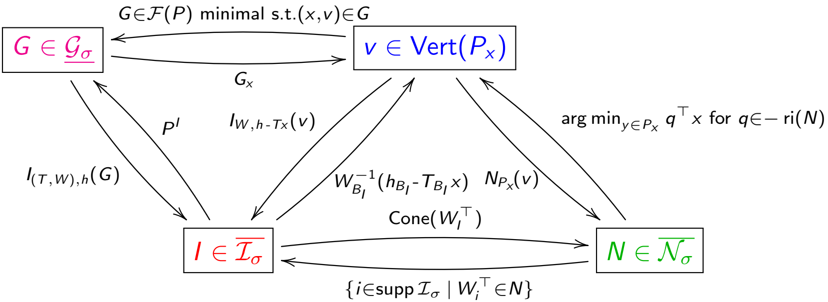
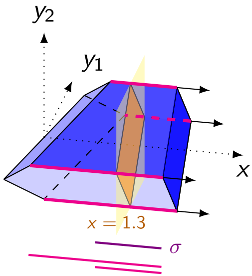
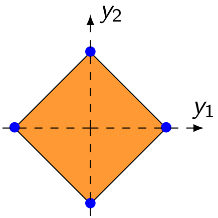
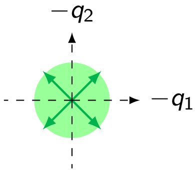




$\overline{\mathcal{I}_\sigma} =$
$\{12, 23, 34, 41\}$





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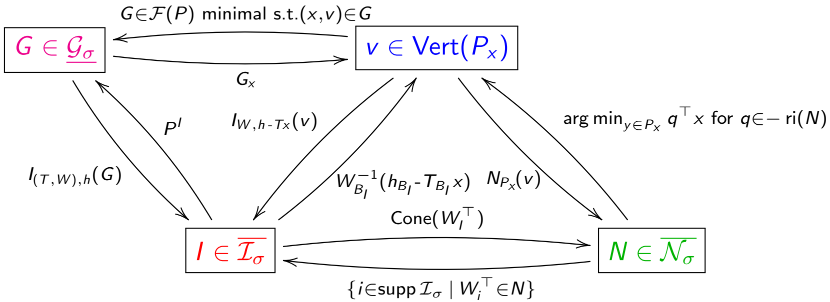
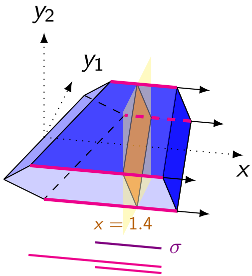
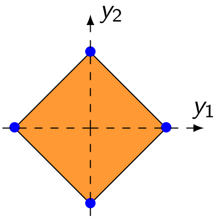
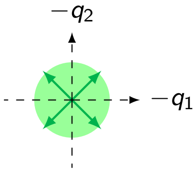




$\overline{\mathcal{I}_\sigma} =$
$\{12, 23, 34, 41\}$





$\mathcal{I}_\sigma$ contains all needed informations

Recall that, for all $x \in \mathrm{ri}(\sigma)$

$$V(x) = \sum_{I \in \overline{\mathcal{I}_\sigma}} \mathbb{E}\left[\mathbf{q}^\top \mathbb{1}_{\mathbf{q} \in -\operatorname{Cone}(W_I^\top)}\right] W_{B_I}^{-1}(h_{B_I} - T_{B_I}x) \quad \text{with } B_I \text{ basis } \subset I$$

Moreover, we can show

$$x \in \mathrm{ri}(\sigma) \iff \begin{cases} \forall I \in \overline{\mathcal{I}_\sigma}, & \\ \forall i \in I \setminus B_I, & v_i^{B_I} x = u_i^{B_I} \\ \forall j \in [q] \setminus I, & v_j^{B_I} x < u_j^{B_I} \end{cases}$$

where

$$v_i^B := T_i - W_i W_B^{-1} T_B$$
$$u_i^B := h_i - W_i W_B^{-1} h_B$$

If σ and τ are adjacent chambers in $\mathcal{C}(P, \pi)$
Then, $\mathcal{I}_\sigma$ and $\mathcal{I}_\tau$ do not differ at lot.

➥ Idea: Pivot between vertices in the chamber complex and update $\mathcal{I}_\sigma$

$\mathcal{I}_\sigma$ contains all needed informations

Recall that, for all $x \in \mathrm{ri}(\sigma)$

$$V(x) = \sum_{I \in \overline{\mathcal{I}_\sigma}} \mathbb{E}\left[\mathbf{q}^\top \mathbb{1}_{\mathbf{q} \in -\operatorname{Cone}(W_I^\top)}\right] W_{B_I}^{-1}(h_{B_I} - T_{B_I} x) \quad \text{with } B_I \text{ basis } \subset I$$

Moreover, we can show

$$x \in \mathrm{ri}(\sigma) \iff \begin{cases} \forall I \in \overline{\mathcal{I}_\sigma}, \\ \forall i \in I \setminus B_I, \quad v_i^{B_I} x = u_i^{B_I} \\ \forall j \in [q] \setminus I, \quad v_j^{B_I} x < u_j^{B_I} \end{cases}$$

where

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$$V(x) = \sum_{I \in \overline{\mathcal{I}_\sigma}} \mathbb{E}\big[\mathbf{q}^\top \mathbb{1}_{\mathbf{q} \in -\operatorname{Cone}(W_I^\top)}\big] W_{B_I}^{-1}(h_{B_I} - T_{B_I} x) \quad \text{with } B_I \text{ basis } \subset I$$

Moreover, we can show

$$x \in \mathrm{ri}(\sigma) \iff \begin{cases} \forall I \in \overline{\mathcal{I}_\sigma}, \\ \forall i \in I \backslash B_I, \quad v_i^{B_I} x = u_i^{B_I} \\ \forall j \in [q] \backslash I, \quad v_j^{B_I} x < u_j^{B_I} \end{cases}$$

where

$$v_i^B := T_i - W_i W_B^{-1} T_B$$
$$u_i^B := h_i - W_i W_B^{-1} h_B$$

If σ and τ are adjacent chambers in $\mathcal{C}(P, \pi)$
Then, $\mathcal{I}_\sigma$ and $\mathcal{I}_\tau$ do not differ at lot.

➥ Idea: Pivot between vertices in the chamber complex and update $\mathcal{I}_\sigma$

Secondary simplex algorithm: pivot procedure

Compute every edges directions d adjacent to x and $\overline{\mathcal{I}_d} := \mathcal{I}\big(W, h - T(x + \varepsilon d)\big)$ for $\varepsilon > 0$ small enough;

if *If there exists an edge with direction d such that,* $c^\top d + \sum_{I \in \overline{\mathcal{I}_d}} \mathbb{E}\big[\mathbf{q}\mathbb{1}_{\mathbf{q} \in -\operatorname{Cone}(W_I^\top)}\big] W_{B_I}^{-1} T_{B_I} d_{B_I} < 0$ **then**

Choose d such a direction and set $\overline{\mathcal{I}} := \overline{\mathcal{I}_d}$;

Let $\lambda = \min_{I \in \overline{\mathcal{I}}, j \in [p] \setminus I | v_j^{B_I} d > 0} \frac{u_j^{B_I} - v_j^{B_I} x}{v_j^{B_I} d}$;

Let $Sat := \{(I, j) \mid I \in \overline{\mathcal{I}},\ j \in [p] \setminus I, \lambda = \frac{u_j^{B_I} - v_j^{B_I} x}{v_j^{B_I} d}\}$;

if $\lambda = +\infty$ **then**

Return "The value of (2SLP) is $-\infty$"

else

Let $\mathcal{I}_{sat} = \{I \in \overline{\mathcal{I}} \mid \exists j, (I, j) \in Sat\}$;

Let $\mathcal{J}_{new} = \{I \cup \bigcup_{j | (I,j) \in Sat} \{j\} \mid I \in \mathcal{I}_{sat}\}$;

Compute $\overline{\mathcal{J}} = (\overline{\mathcal{I}} \setminus \mathcal{I}_{sat}) \cup \mathcal{J}_{new}$;

Return $(x + \lambda d, \overline{\mathcal{J}})$

end

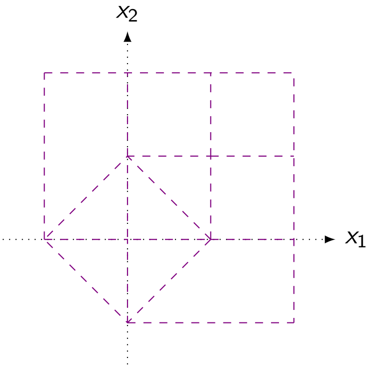
else

Return "x is an optimal solution"

end

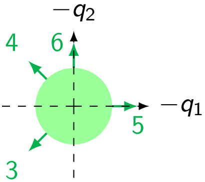
Simplex for 2SLP

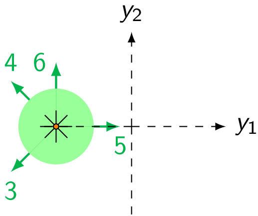
$$
\begin{aligned}
y_1 + y_2 &\leqslant 1 && (1)\\
y_1 - y_2 &\leqslant 1 && (2)\\
-y_1 - y_2 &\leqslant 1 && (3)\\
-y_1 + y_2 &\leqslant 1 && (4)\\
y_1 &\leqslant x_1 && (5)\\
y_2 &\leqslant x_2 && (6)\\
x_1 &\leqslant 2 && (7)\\
x_2 &\leqslant 2 && (8)
\end{aligned}
$$

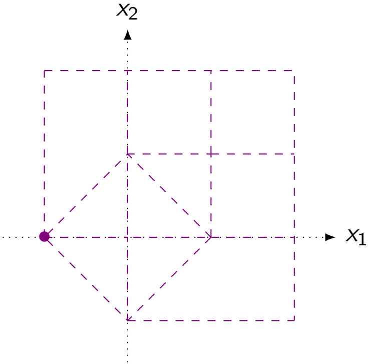


Simplex for 2SLP

$\overline{\mathcal{I}} = \{3456\}$

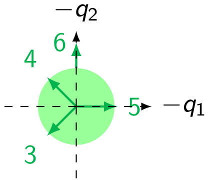


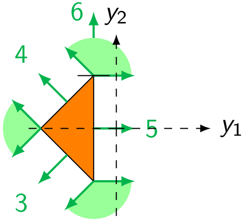


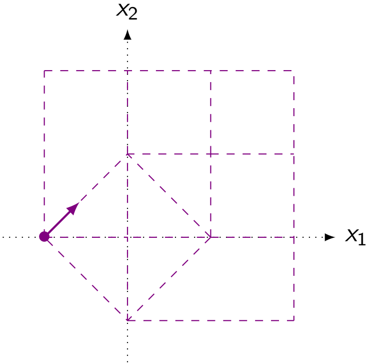


Simplex for 2SLP

$\overline{\mathcal{I}} = \{34, 35, 456\}$

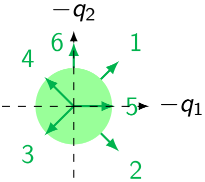


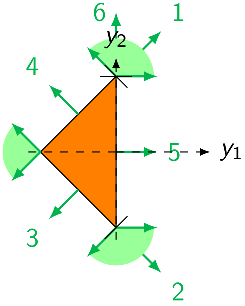


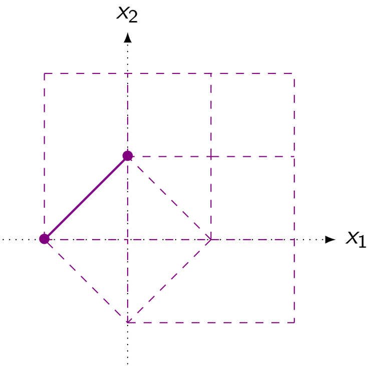


Simplex for 2SLP

$\overline{\mathcal{I}} = \{34, 235, 1456\}$

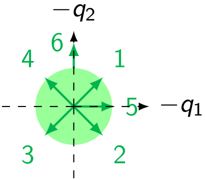


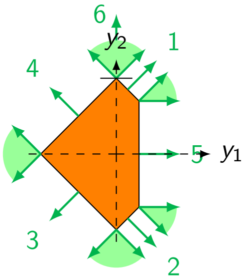


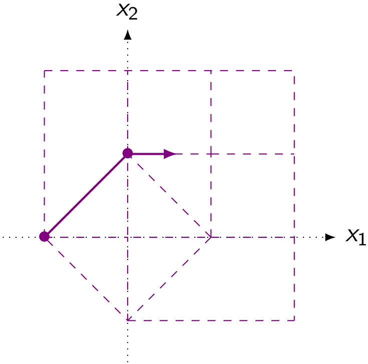


Simplex for 2SLP

$$\overline{\mathcal{I}} = \{34, 23, 25, 146, 15\}$$

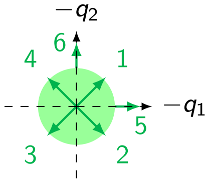


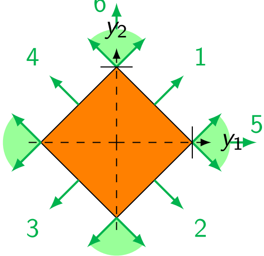


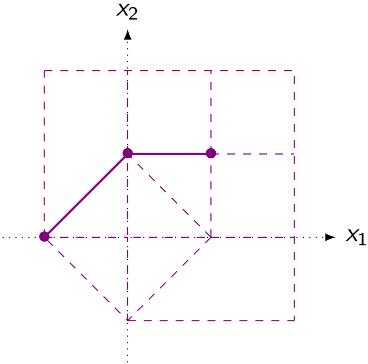


Simplex for 2SLP

$\overline{\mathcal{I}} = \{34, 23, 125, 146\}$

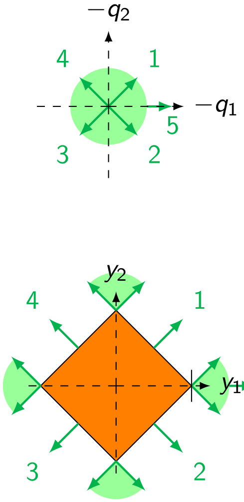


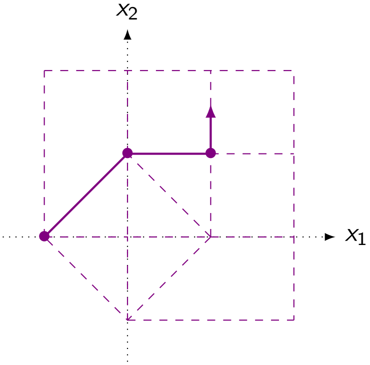




Simplex for 2SLP

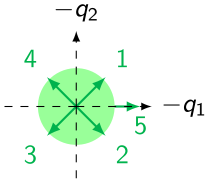
$\overline{\mathcal{I}} = \{34, 23, 125, 14\}$

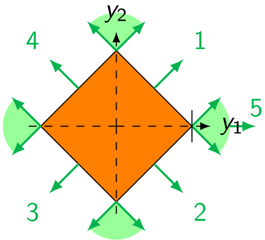


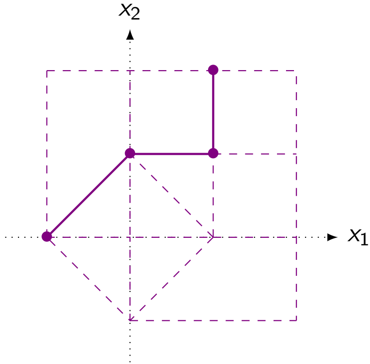


Simplex for 2SLP

$$\overline{\mathcal{I}} = \{348, 238, 1258, 148\}$$

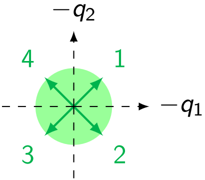


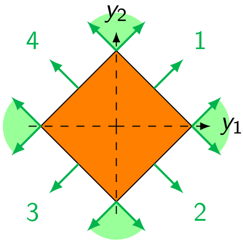


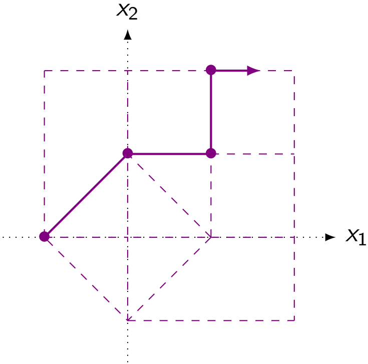


Simplex for 2SLP

$\overline{\mathcal{I}} = \{348, 238, 128, 148\}$

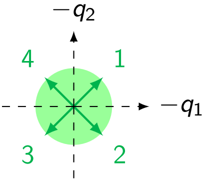


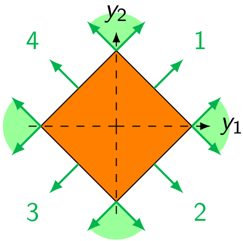


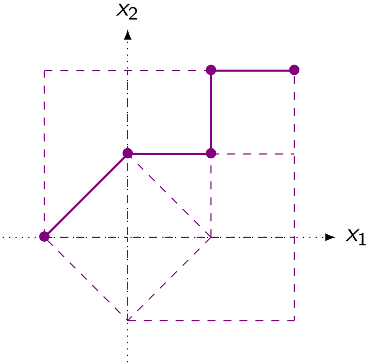


Simplex for 2SLP

$\overline{\mathcal{I}} = \{3478, 2378, 1278, 1478\}$







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Thank you for listening ! Any question ?