2-Stage Stochastic Linear Problem and Polyhedral Geometry

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Contents

Linear Programming

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & c^\top x \\ \text{s.t.} \quad & Ax \leqslant b \end{aligned}$$

Example: $P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$

$$A = \begin{pmatrix} 1 & 1 \\ & \\ & \\ & \\ & \\ & \\ & \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ \\ \\ \\ \\ \\ \\ \end{pmatrix}$$

$x_1 + x_2 \leqslant 1$ (1)

(2)

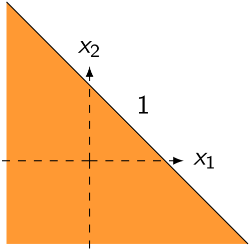
(3)

(4)

(5)

(6)

(7)



Linear Programming

$$\min_{x \in \mathbb{R}^n} \quad c^\top x$$
$$\text{s.t.} \quad Ax \leqslant b$$

Example: $P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ & \\ & \\ & \\ & \\ & \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 1 \\ \\ \\ \\ \\ \\ \end{pmatrix}$$

$x_1 + x_2 \leqslant 1$ (1)

$x_1 - x_2 \leqslant 1$ (2)

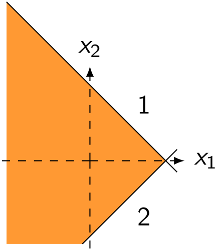
(3)

(4)

(5)

(6)

(7)



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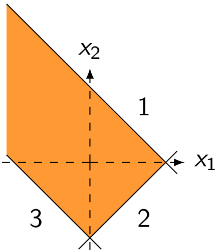
Example: $P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -1 & -1 \\ & \\ & \\ & \\ & \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \\ \\ \\ \\ \end{pmatrix}$$

$$\begin{aligned} x_1 + x_2 &\leqslant 1 && (1) \\ x_1 - x_2 &\leqslant 1 && (2) \\ -x_1 - x_2 &\leqslant 1 && (3) \\ & && (4) \\ & && (5) \\ & && (6) \\ & && (7) \end{aligned}$$



Linear Programming

$$\min_{x \in \mathbb{R}^n} \quad c^\top x$$
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Example: $P = \{x \in \mathbb{R}^n \,|\, Ax \leqslant b\}$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -1 & -1 \\ -1 & 1 \\ & \\ & \\ & \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ \\ \\ \\ \end{pmatrix}$$

$$x_1 + x_2 \leqslant 1 \quad (1)$$
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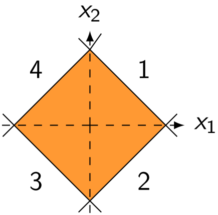
$$(7)$$



Linear Programming

$$\min_{x \in \mathbb{R}^n} \quad c^\top x$$
$$\text{s.t.} \quad Ax \leqslant b$$

Example: $P = \{x \in \mathbb{R}^n \,|\, Ax \leqslant b\}$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -1 & -1 \\ -1 & 1 \\ 1 & 0 \\ & \\ & \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0.5 \\ \\ \end{pmatrix}$$

$$x_1 + x_2 \leqslant 1 \quad (1)$$
$$x_1 - x_2 \leqslant 1 \quad (2)$$
$$-x_1 - x_2 \leqslant 1 \quad (3)$$
$$-x_1 + x_2 \leqslant 1 \quad (4)$$
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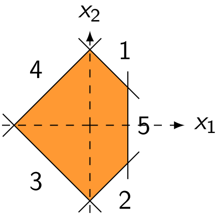


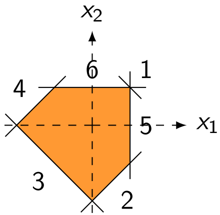
Linear Programming

$$\min_{x \in \mathbb{R}^n} \quad c^\top x$$
$$\text{s.t.} \quad Ax \leqslant b$$

Example: $P = \{x \in \mathbb{R}^n \,|\, Ax \leqslant b\}$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -1 & -1 \\ -1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0.5 \\ 0.5 \end{pmatrix}$$

$$\begin{aligned} x_1 + x_2 &\leqslant 1 && (1) \\ x_1 - x_2 &\leqslant 1 && (2) \\ -x_1 - x_2 &\leqslant 1 && (3) \\ -x_1 + x_2 &\leqslant 1 && (4) \\ x_1 &\leqslant 0.5 && (5) \\ x_2 &\leqslant 0.5 && (6) \\ & && (7) \end{aligned}$$



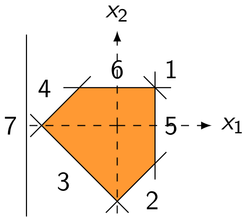
Linear Programming

$$\min_{x \in \mathbb{R}^n} \quad c^\top x$$
$$\text{s.t.} \quad Ax \leqslant b$$

Example: $P = \{x \in \mathbb{R}^n \,|\, Ax \leqslant b\}$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -1 & -1 \\ -1 & 1 \\ 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0.5 \\ 0.5 \\ -1.2 \end{pmatrix}$$

$$\begin{aligned} x_1 + x_2 &\leqslant 1 && (1) \\ x_1 - x_2 &\leqslant 1 && (2) \\ -x_1 - x_2 &\leqslant 1 && (3) \\ -x_1 + x_2 &\leqslant 1 && (4) \\ x_1 &\leqslant 0.5 && (5) \\ x_2 &\leqslant 0.5 && (6) \\ x_1 &\geqslant -1.2 && (7) \end{aligned}$$



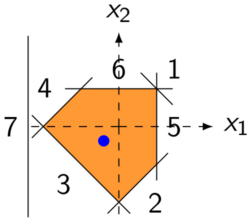
Active constraints

Definition

We denote by $\mathcal{I}(A,b)$, the collection of sets of active constraints as :

$$\mathcal{I}(A,b) = \{I_{A,b}(x) \mid Ax \leqslant b\}$$

with $I_{A,b}(x) := \{i \in [q] \mid A_i x = b_i\}$



$P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$

$I_{A,b}(x) = \emptyset$

To ease the notation, we write:

$$\mathcal{I}(A,b) = \{\emptyset, \qquad \}$$

Active constraints

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We denote by $\mathcal{I}(A, b)$, the collection of sets of active constraints as :

$$\mathcal{I}(A, b) = \{ I_{A,b}(x) \mid Ax \leqslant b \}$$

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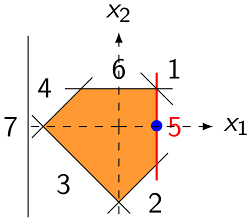

$P = \{ x \in \mathbb{R}^n \mid Ax \leqslant b \}$

$I_{A,b}(x) = \{5\}$

To ease the notation, we write:

$\mathcal{I}(A, b) = \{\emptyset, 5, \qquad\qquad \}$

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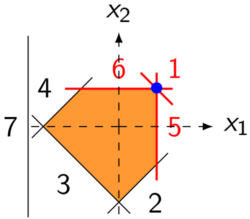
We denote by $\mathcal{I}(A, b)$, the collection of sets of active constraints as :

$$\mathcal{I}(A, b) = \{I_{A,b}(x) \mid Ax \leqslant b\}$$

with $I_{A,b}(x) := \{i \in [q] \mid A_i x = b_i\}$



$P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$

$I_{A,b}(x) = \{1, 5, 6\}$

To ease the notation, we write:

$$\mathcal{I}(A, b) = \{\emptyset, 5, 156, \qquad \}$$

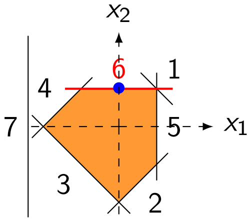
Active constraints

Definition

We denote by $\mathcal{I}(A, b)$*, the collection of sets of active constraints as :*

$$\mathcal{I}(A, b) = \{I_{A,b}(x) \mid Ax \leqslant b\}$$

with $I_{A,b}(x) := \{i \in [q] \mid A_i x = b_i\}$



$P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$

$I_{A,b}(x) = \{6\}$

To ease the notation, we write:

$$\mathcal{I}(A, b) = \{\emptyset, 5, 156, 6, \qquad \}$$

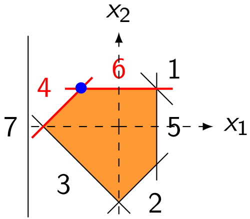
Active constraints

Definition

We denote by $\mathcal{I}(A,b)$*, the collection of sets of active constraints as :*

$$\mathcal{I}(A,b) = \{I_{A,b}(x) \mid Ax \leqslant b\}$$

with $I_{A,b}(x) := \{i \in [q] \mid A_i x = b_i\}$



$P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$

$I_{A,b}(x) = \{4, 6\}$

To ease the notation, we write:

$$\mathcal{I}(A,b) = \{\emptyset, 5, 156, 6, 46, \qquad\qquad \}$$

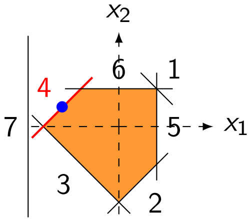
Active constraints

Definition

We denote by $\mathcal{I}(A,b)$, the collection of sets of active constraints as :

$$\mathcal{I}(A,b) = \{I_{A,b}(x) \mid Ax \leqslant b\}$$

with $I_{A,b}(x) := \{i \in [q] \mid A_i x = b_i\}$



$P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$

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To ease the notation, we write:

$$\mathcal{I}(A,b) = \{\emptyset, 5, 156, 6, 46, 4, \qquad \}$$

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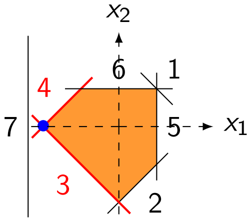

$P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$

$I_{A,b}(x) = \{3, 4\}$

To ease the notation, we write:

$$\mathcal{I}(A, b) = \{\emptyset, 5, 156, 6, 46, 4, 34, \qquad \}$$

Active constraints

Definition

We denote by $\mathcal{I}(A, b)$, the collection of sets of active constraints as :

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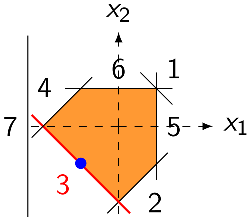
with $I_{A,b}(x) := \{i \in [q] \mid A_i x = b_i\}$



$P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$

$I_{A,b}(x) = \{3\}$

To ease the notation, we write:

$$\mathcal{I}(A, b) = \{\emptyset, 5, 156, 6, 46, 4, 34, 3, \qquad \}$$

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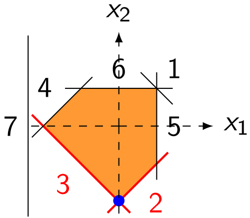


$P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$

$I_{A,b}(x) = \{2, 3\}$

To ease the notation, we write:

$$\mathcal{I}(A, b) = \{\emptyset, 5, 156, 6, 46, 4, 34, 3, 23, \qquad \}$$

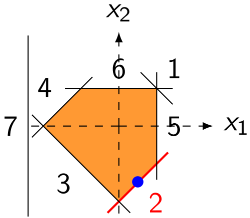
Active constraints

Definition

We denote by $\mathcal{I}(A, b)$, the collection of sets of active constraints as :

$$\mathcal{I}(A, b) = \{I_{A,b}(x) \mid Ax \leqslant b\}$$

with $I_{A,b}(x) := \{i \in [q] \mid A_i x = b_i\}$



$P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$

$I_{A,b}(x) = \{2\}$

To ease the notation, we write:

$$\mathcal{I}(A, b) = \{\emptyset, 5, 156, 6, 46, 4, 34, 3, 23, 2, \quad \}$$

Active constraints

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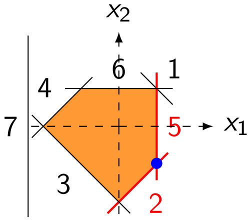


$P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$

$I_{A,b}(x) = \{2, 5\}$

To ease the notation, we write:

$$\mathcal{I}(A, b) = \{\emptyset, 5, 156, 6, 46, 4, 34, 3, 23, 2, 25\}$$

Faces

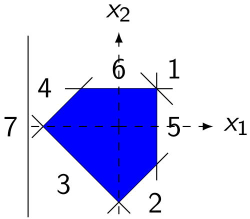
Definition

Let $I \in \mathcal{I}(A, b)$, we denote by P^I the face of P such that:

$$P^I = \{x \in P \,|\, A_I x = b_I\}$$

We have $\dim(P^I) = n - \text{rg}(A_I)$

Example for $I = \emptyset$



P, P^I

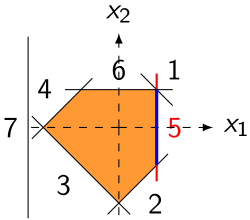
Faces

Definition

Let $I \in \mathcal{I}(A, b)$, we denote by P^I the face of P such that:

$$P^I = \{x \in P \mid A_I x = b_I\}$$

We have $\dim(P^I) = n - \mathrm{rg}(A_I)$
Example for $I = \{5\}$



P, P^I

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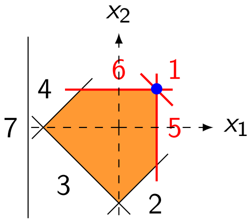
Example for $I = \{1, 5, 6\}$



P, P^I

Faces

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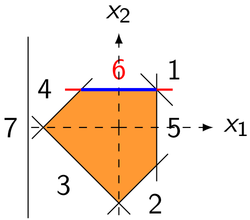
$$P^I = \{x \in P \mid A_I x = b_I\}$$

We have $\dim(P^I) = n - \text{rg}(A_I)$

Example for $I = \{6\}$



P, P^I

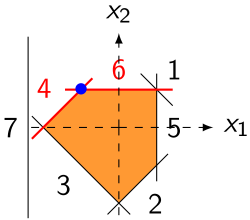
Faces

Definition

Let $I \in \mathcal{I}(A, b)$, we denote by P^I the face of P such that:

$$P^I = \{x \in P \mid A_I x = b_I\}$$

We have $\dim(P^I) = n - \text{rg}(A_I)$
Example for $I = \{4, 6\}$



P, P^I

Faces

Definition

Let $I \in \mathcal{I}(A, b)$, we denote by P^I the face of P such that:

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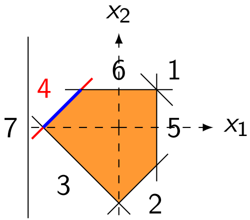
We have $\dim(P^I) = n - \text{rg}(A_I)$

Example for $I = \{4\}$



P, P^I

Faces

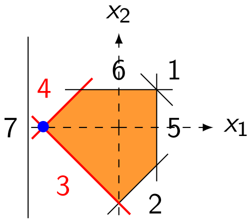
Definition

Let $I \in \mathcal{I}(A, b)$, we denote by P^I the face of P such that:

$$P^I = \{x \in P \mid A_I x = b_I\}$$

We have $\dim(P^I) = n - \text{rg}(A_I)$

Example for $I = \{3, 4\}$



P, P^I

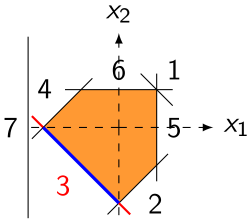
Faces

Definition

Let $I \in \mathcal{I}(A, b)$, we denote by P^I the face of P such that:

$$P^I = \{x \in P \,|\, A_I x = b_I\}$$

We have $\dim(P^I) = n - \mathrm{rg}(A_I)$
Example for $I = \{3\}$



P, P^I

Faces

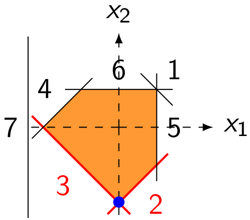
Definition

Let $I \in \mathcal{I}(A, b)$, we denote by P^I the face of P such that:

$$P^I = \{x \in P \mid A_I x = b_I\}$$

We have $\dim(P^I) = n - \mathrm{rg}(A_I)$

Example for $I = \{2, 3\}$



P, P^I

Faces

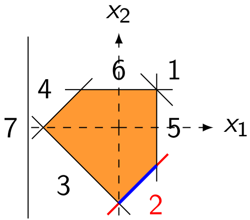
Definition

Let $I \in \mathcal{I}(A, b)$, we denote by P^I the face of P such that:

$$P^I = \{x \in P \mid A_I x = b_I\}$$

We have $\dim(P^I) = n - \text{rg}(A_I)$

Example for $I = \{2\}$



P, P^I

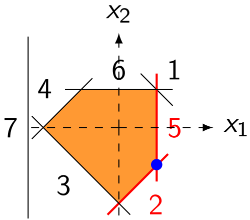
Faces

Definition

Let $I \in \mathcal{I}(A, b)$, we denote by P^I the face of P such that:

$$P^I = \{x \in P \mid A_I x = b_I\}$$

We have $\dim(P^I) = n - \mathrm{rg}(A_I)$
Example for $I = \{2, 5\}$



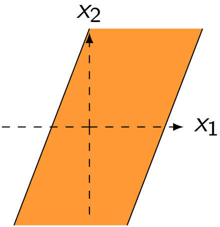
P, P^I

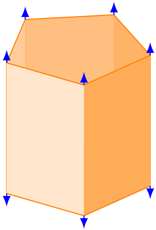
Polyhedra without any vertex ?

Definition (Lineality space)

$\mathrm{Lin}(C) := \{u \in C \mid \forall t \in \mathbb{R},\ \forall x \in c,\ x + tu \in C\}.$

$$\mathrm{Lin}\left(\{x \mid Ax \leqslant b\}\right) = \mathrm{Ker}(A)$$





Bases and Vertices

Let $P = \{x \in \mathbb{R}^n | Ax \leqslant b\}$ with $A \in \mathbb{R}^{p \times n}$ and $b \in \mathbb{R}^p$.

Definition

A basis B is a subset of $[p]$ such that $A_B = (A_{i,j})_{i \in B, 1 \leqslant j \leqslant n}$ is invertible.
A vertex of P is a face of dimension 0. $\mathrm{Vert}(P)$ is the set of vertices.

$$\begin{aligned} \mathrm{Vert}(P) \neq \emptyset &\iff A \text{ admits at least one basis} \\ &\iff \mathrm{rg}(A) = n \\ &\iff \mathrm{Lin}(P) = \{0\} \end{aligned}$$

Under this assumption,
For every $I \in \overline{\mathcal{I}(A, b)}$, we can extract a basis B_I and $P^I = \{A_{B_I}^{-1} b_{B_I}\}$.
If $c \notin \mathrm{Lin}(P)^\perp = \mathrm{Im}(A^\top)$, $\min_{x \in P} c^\top x = -\infty$.
Otherwise, we can write $P = P_0 + \mathrm{Lin}(P)$ with $\mathrm{Lin}(P_0) = \{0\}$:
We make this assumption without loss of generality

Bases and Vertices

Let $P = \{x \in \mathbb{R}^n | Ax \leqslant b\}$ with $A \in \mathbb{R}^{p \times n}$ and $b \in \mathbb{R}^p$.

Definition

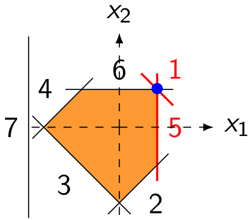
A basis B is a subset of $[p]$ such that $A_B = (A_{i,j})_{i \in B, 1 \leqslant j \leqslant n}$ is invertible.
A vertex of P is a face of dimension 0. $\mathrm{Vert}(P)$ is the set of vertices.

$$\begin{aligned} \mathrm{Vert}(P) \neq \emptyset &\iff A \text{ admits at least one basis} \\ &\iff \mathrm{rg}(A) = n \\ &\iff \mathrm{Lin}(P) = \{0\} \end{aligned}$$

Under this assumption,
For every $I \in \overline{\mathcal{I}(A, b)}$, we can extract a basis B_I and $P^I = \{A_{B_I}^{-1} b_{B_I}\}$.
If $c \notin \mathrm{Lin}(P)^{\perp} = \mathrm{Im}(A^{\top})$, $\min_{x \in P} c^{\top} x = -\infty$.
Otherwise, we can write $P = P_0 + \mathrm{Lin}(P)$ with $\mathrm{Lin}(P_0) = \{0\}$:
We make this assumption without loss of generality

Simplex method

Geometrically:
follow a path on the polyhedron from vertex to vertex



Combinatorially:
pivoting from basis to basis

$$B_1 = \{1, 5\}$$

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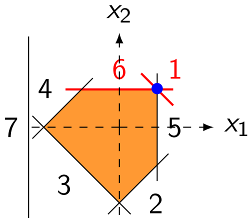

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$$B_2 = \{1, 6\}$$

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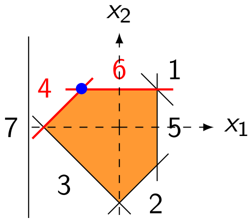

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$$B_3 = \{4, 6\}$$

Simplex method

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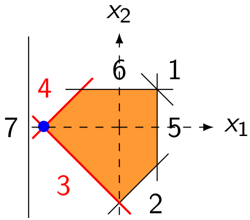

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$$B_1 = \{1, 5\}$$
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$$B_3 = \{4, 6\}$$
$$B_2 = \{3, 4\}$$

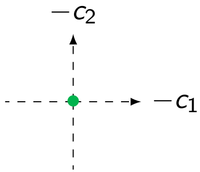
Normal fan $\mathcal{N}(P)$

Definition

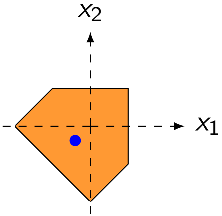
The normal fan of the polyhedron P is

$$\mathcal{N}(P) := \{N_P(x) \,|\, x \in P\}$$

with $N_P(x) = \{c \,|\, \forall x' \in P,\ c^\top(x' - x) \leqslant 0\}$ *the normal cone of P on x.*



$N_P(y)$



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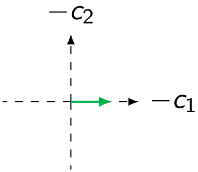

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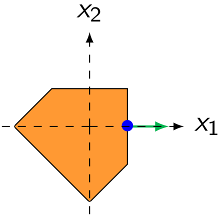

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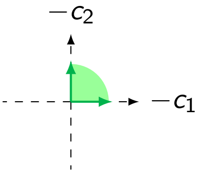

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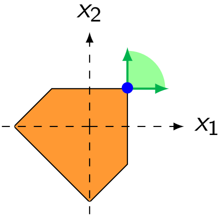

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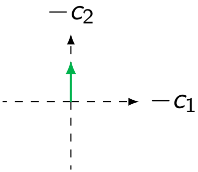

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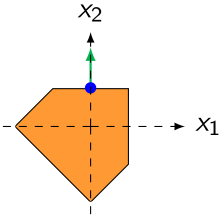

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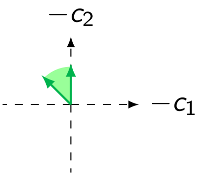

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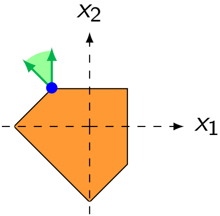

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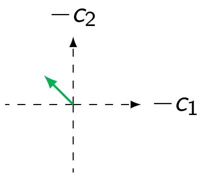

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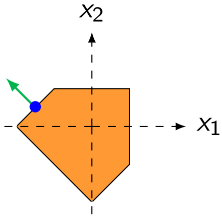

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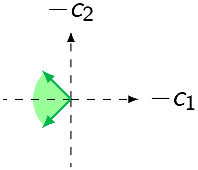

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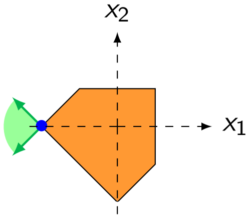

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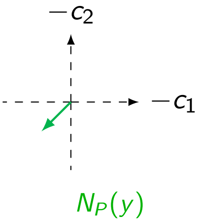

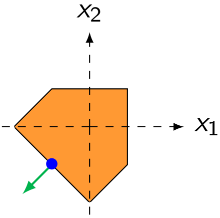

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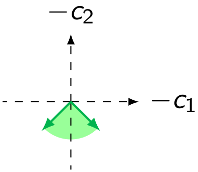

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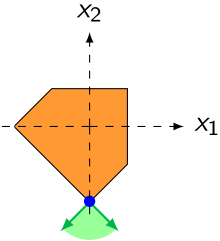

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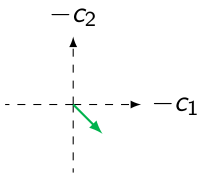

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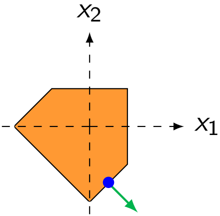

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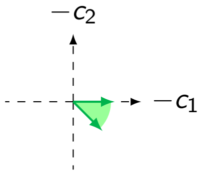

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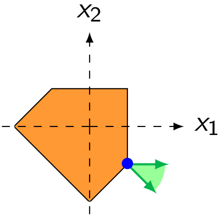

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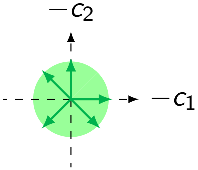
Proposition

$\{\mathrm{ri}(N) \,|\, N \in \mathcal{N}(P)\}$ *is a partition of* $\mathrm{supp}\,\mathcal{N}(P)$ *($= \mathbb{R}^m$ if P is bounded).*



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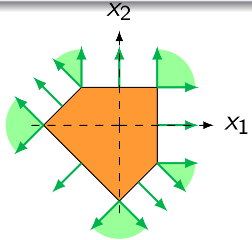


P and $\mathcal{N}(P)$

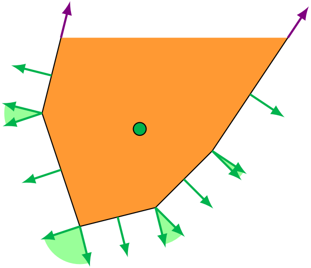
Definition (Recession cone)

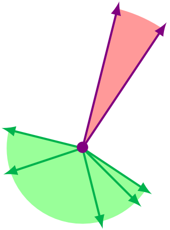
$\mathrm{rc}(C) := \{u \in C \mid \forall t \in \mathbb{R}_+,\ \forall x \in c,\ x + tu \in C\}.$

Let $P = \{x \mid Ax \leqslant b\}$

$$\mathrm{rc}(P) = \{u \mid Au \leqslant 0\}$$

$$-\infty < \begin{cases} \inf_{x\in\mathbb{R}^n} & c^\top x \\ \text{s.t.} & Ax \leqslant b \end{cases} \iff -c \in \mathrm{rc}(P)^\star = \mathrm{Cone}(A^\top) = \mathrm{supp}\left(\mathcal{N}(P)\right)$$

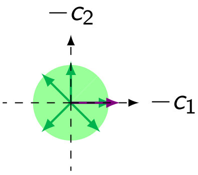




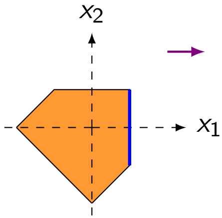
$\mathcal{N}(P)$: partition of cost coherent with the min

For any $N \in \mathcal{N}(P)$ and $-c \to \arg\min_{x \in P} c^\top x$ is constant for all $-c \in \mathrm{ri}(N)$.

$\arg\min_{x \in P} c^\top x$ is a face of P.



Cost $-c$ and $\mathcal{N}(P)$



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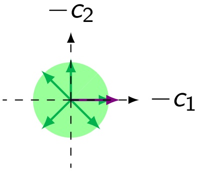

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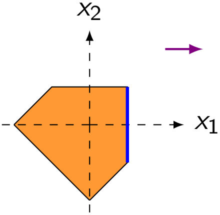

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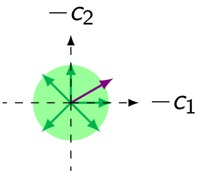

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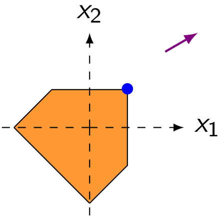

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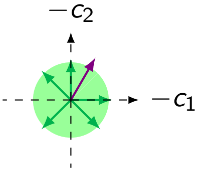

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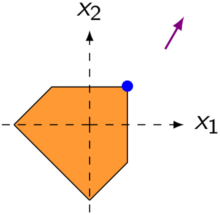

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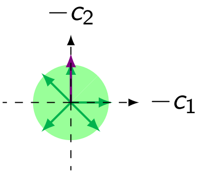

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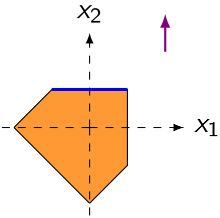

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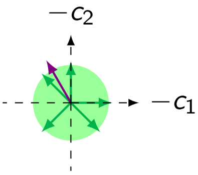

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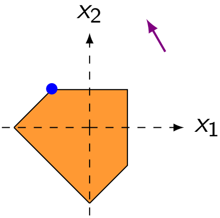

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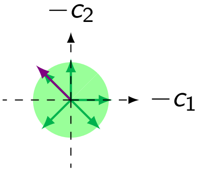

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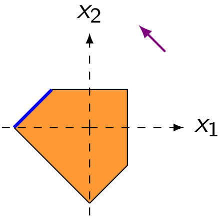

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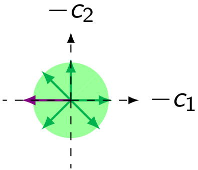

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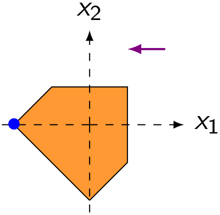

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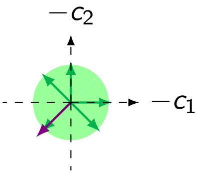
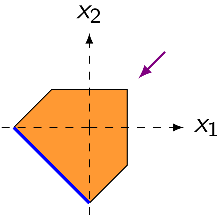


Cost $-c$ and $\mathcal{N}(P)$

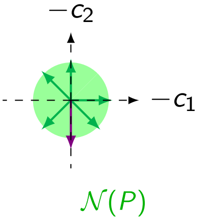


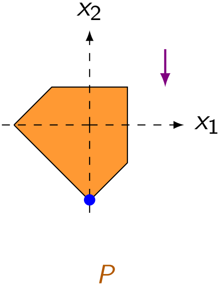
P

$\mathcal{N}(P)$: partition of cost coherent with the min

For any $N \in \mathcal{N}(P)$ and $-c \to \arg\min_{x \in P} c^\top x$ is constant for all $-c \in \mathrm{ri}(N)$.

$\arg\min_{x \in P} c^\top x$ is a face of P.





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For any $N \in \mathcal{N}(P)$ and $-c \to \arg\min_{x \in P} c^\top x$ is constant for all $-c \in \mathrm{ri}(N)$.

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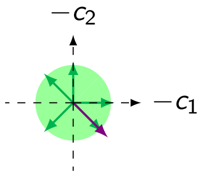
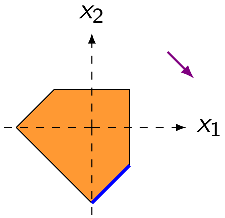


$\mathcal{N}(P)$



P

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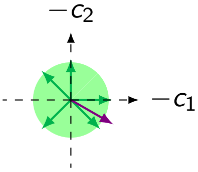
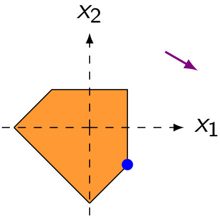


$\mathcal{N}(P)$

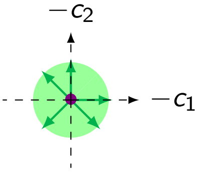


P

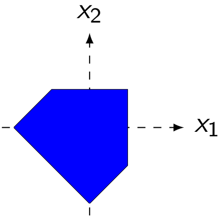
$\mathcal{N}(P)$: partition of cost coherent with the min

For any $N \in \mathcal{N}(P)$ and $-c \to \arg\min_{x \in P} c^\top x$ is constant for all $-c \in \text{ri}(N)$.

$\arg\min_{x \in P} c^\top x$ is a face of P.



$\mathcal{N}(P)$

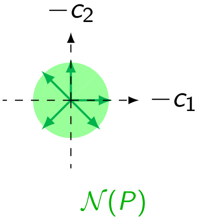


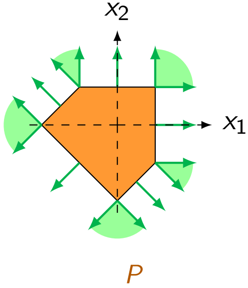
P

$\mathcal{N}(P)$: partition of cost coherent with the min

For any $N \in \mathcal{N}(P)$ and $-c \rightarrow \arg\min_{x \in P} c^\top x$ is constant for all $-c \in \mathrm{ri}(N)$.

$\arg\min_{x \in P} c^\top x$ is a face of P.





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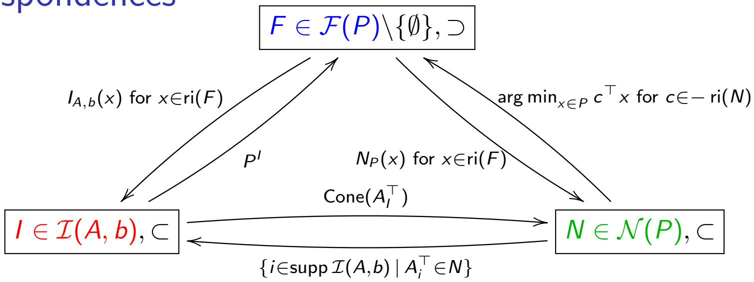
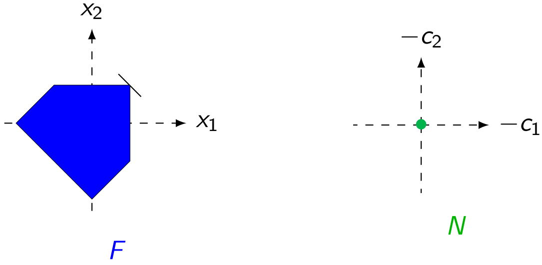


$$\mathrm{rg}(A_I) = n - \dim(F) = \dim(N)$$

$I = \emptyset$



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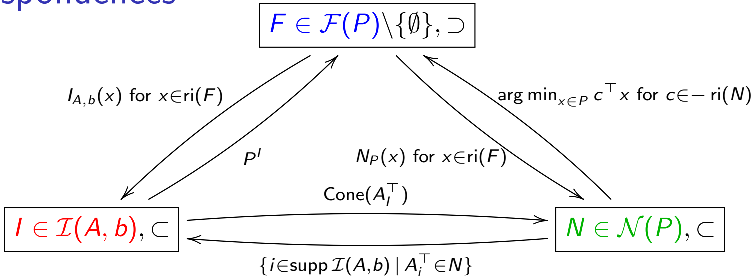
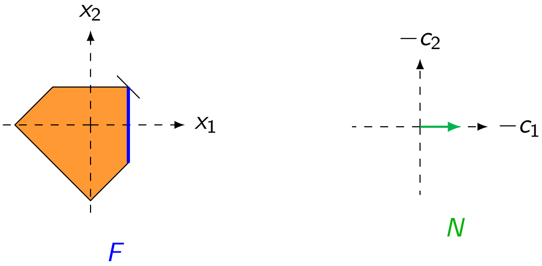


$$\mathrm{rg}(A_I) = n - \dim(F) = \dim(N)$$

$I = \{5\}$



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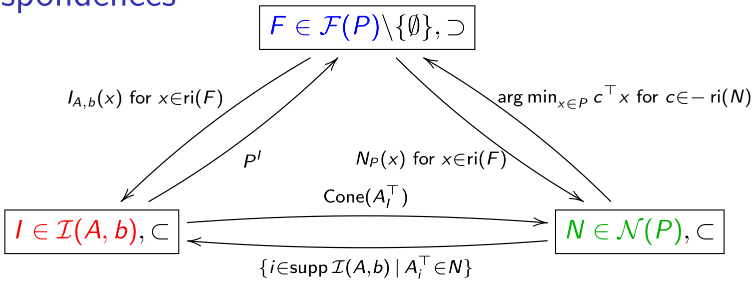
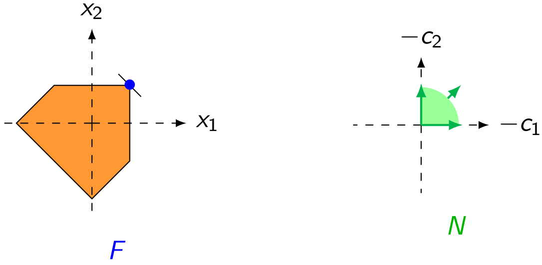


$$\mathrm{rg}(A_I) = n - \dim(F) = \dim(N)$$

$I = \{1, 5, 6\}$



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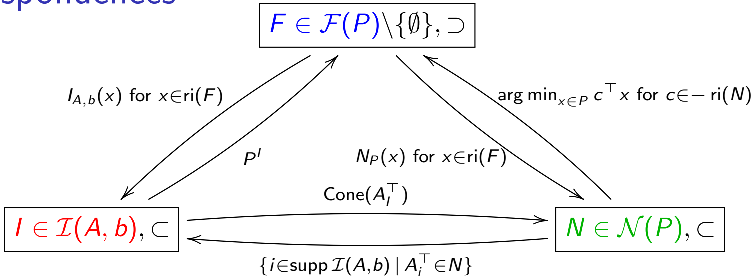
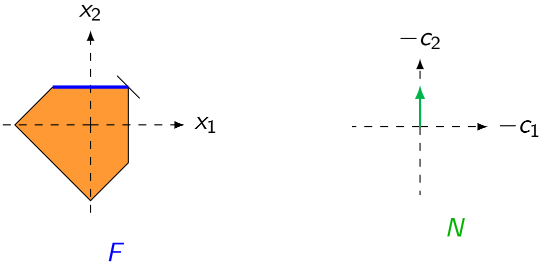


$$\mathrm{rg}(A_I) = n - \dim(F) = \dim(N)$$

$I = \{6\}$



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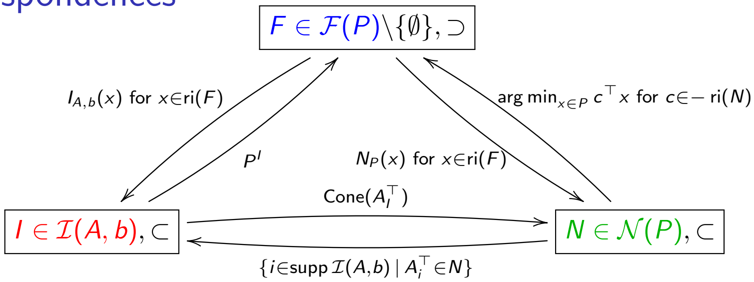
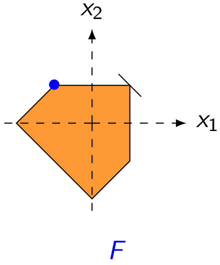
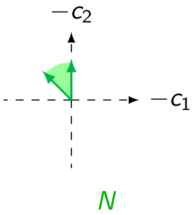


$$\mathrm{rg}(A_I) = n - \dim(F) = \dim(N)$$

$I = \{4, 6\}$





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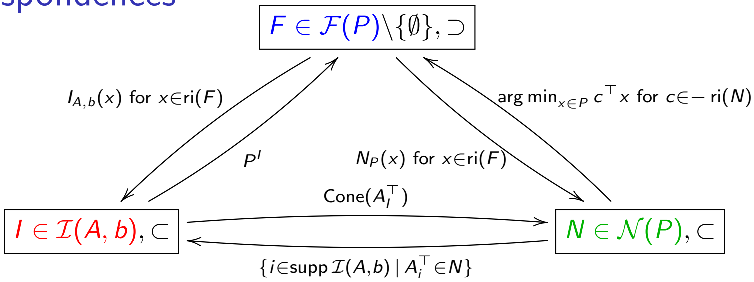
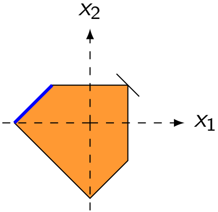
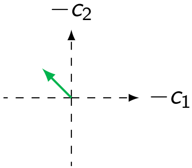


$$\mathrm{rg}(A_I) = n - \dim(F) = \dim(N)$$

$I = \{4\}$

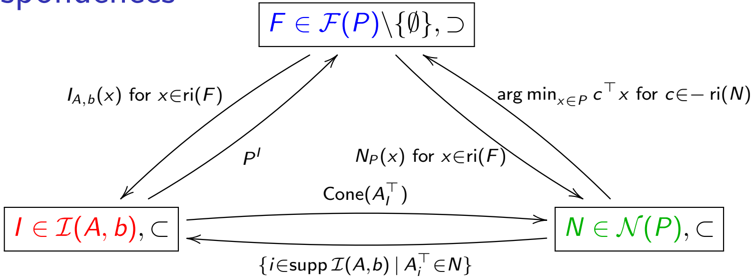


F



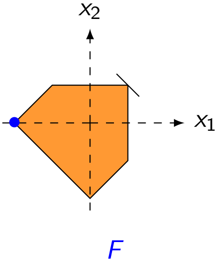
N

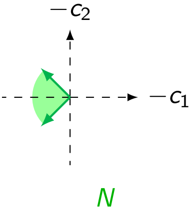
Correspondences



$$\mathrm{rg}(A_I) = n - \dim(F) = \dim(N)$$

$I = \{3, 4\}$





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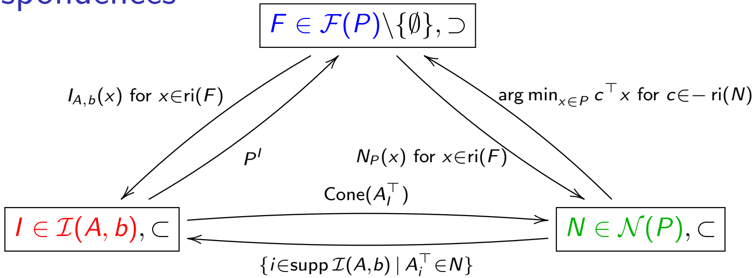
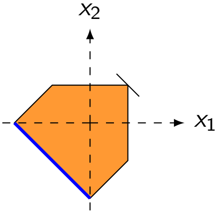
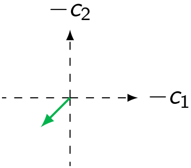


$$\mathrm{rg}(A_I) = n - \dim(F) = \dim(N)$$

$I = \{3\}$

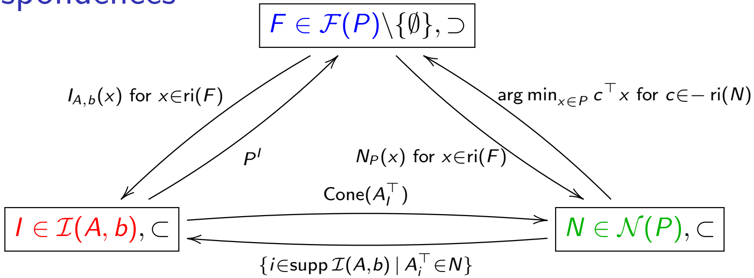


F



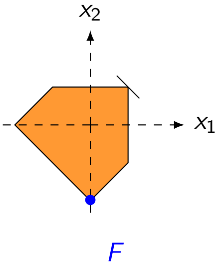
N

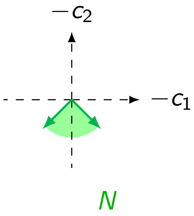
Correspondences



$$\mathrm{rg}(A_I) = n - \dim(F) = \dim(N)$$

$I = \{2, 3\}$





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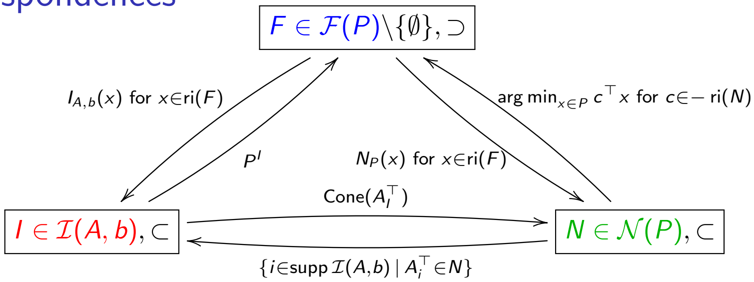
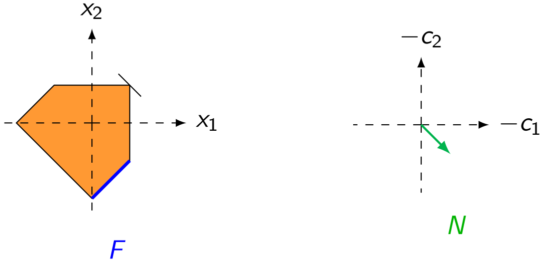


$$\mathrm{rg}(A_I) = n - \dim(F) = \dim(N)$$

$I = \{2\}$



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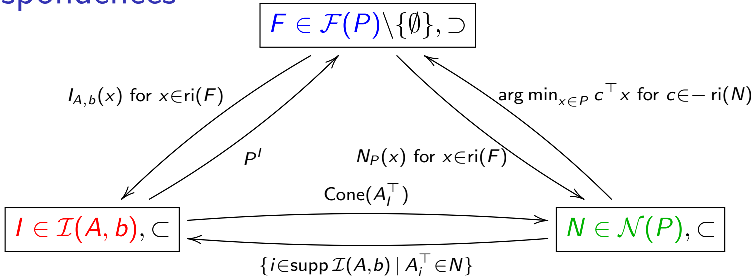
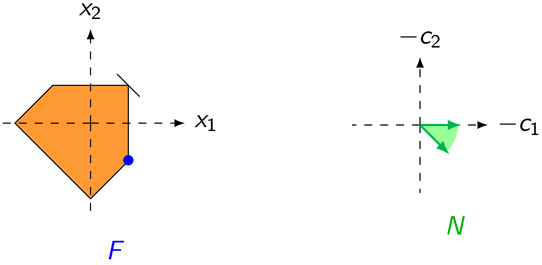
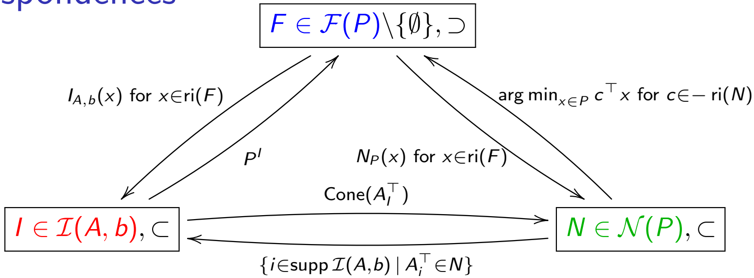


$$\mathrm{rg}(A_I) = n - \dim(F) = \dim(N)$$

$I = \{2, 5\}$

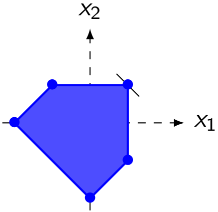


Correspondences

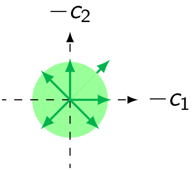


$$\mathrm{rg}(A_I) = n - \dim(F) = \dim(N)$$

$\mathcal{I}(A, b) =$
$\{\emptyset, 5, 156, 6, 46, 4,$
$34, 3, 23, 2, 25\}$

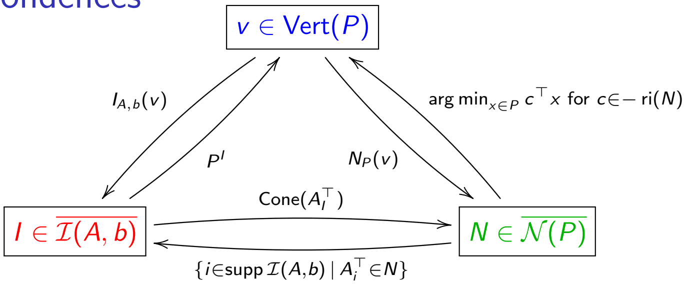


$\mathcal{F}(P)$

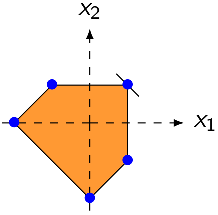


$\mathcal{N}(P)$

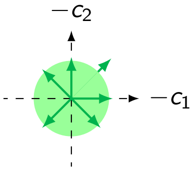
Correspondences



$\overline{\mathcal{I}(A,b)} =$
$\{156, 46, 34, 23, 25\}$



$\mathrm{Vert}(P)$



$\overline{\mathcal{N}(P)}$

Contents

2-Stage Stochastic Linear Programming

$$\min_{x \in \mathbb{R}^n} \quad c^\top x + \mathbb{E}\left[\begin{array}{ll} \min\limits_{y \in \mathbb{R}^m} & \mathbf{q}^\top y \\ \text{s.t.} & Tx + Wy \leqslant h \end{array}\right] \qquad \text{(2SLP)}$$

$$\text{s.t.} \quad Ax \leqslant b$$

where $T \in \mathbb{R}^{p \times n}$, $W \in \mathbb{R}^{p \times m}$ and $h \in \mathbb{R}^p$.
We can assume $A = 0$ and $b = 0$:
We set

$$\widetilde{T} := \begin{pmatrix} T \\ A \end{pmatrix}, \quad \widetilde{W} := \begin{pmatrix} W \\ 0 \end{pmatrix} \quad \text{and } \widetilde{h} = \begin{pmatrix} h \\ b \end{pmatrix}$$

2-Stage Stochastic Linear Programming

$$\min_{x \in \mathbb{R}^n} \quad c^\top x + \mathbb{E}\begin{bmatrix} \min\limits_{y \in \mathbb{R}^m} & \mathbf{q}^\top y & \\ \text{s.t.} & Tx + Wy & \leqslant h \\ & Ax & \leqslant b \end{bmatrix}$$

where $T \in \mathbb{R}^{p \times n}, W \in \mathbb{R}^{p \times m}$ and $h \in \mathbb{R}^p$.
We can assume $A = 0$ and $b = 0$:
We set

$$\widetilde{T} := \begin{pmatrix} T \\ A \end{pmatrix}, \quad \widetilde{W} := \begin{pmatrix} W \\ 0 \end{pmatrix} \quad \text{and } \widetilde{h} = \begin{pmatrix} h \\ b \end{pmatrix}$$

2-Stage Stochastic Linear Programming

$$\min_{x\in\mathbb{R}^n} \quad c^\top x + \mathbb{E}\left[\begin{array}{ll} \min\limits_{y\in\mathbb{R}^m} & \mathbf{q}^\top y \\ \text{s.t.} & Tx + Wy \leqslant h \\ & Ax + 0y \leqslant b \end{array}\right]$$

where $T \in \mathbb{R}^{p\times n}, W \in \mathbb{R}^{p\times m}$ and $h \in \mathbb{R}^p$.
We can assume $A = 0$ and $b = 0$:
We set

$$\widetilde{T} := \begin{pmatrix} T \\ A \end{pmatrix}, \quad \widetilde{W} := \begin{pmatrix} W \\ 0 \end{pmatrix} \quad \text{and } \widetilde{h} = \begin{pmatrix} h \\ b \end{pmatrix}$$

2-Stage Stochastic Linear Programming

$$\min_{x\in\mathbb{R}^n} \quad c^\top x + \mathbb{E}\begin{bmatrix} \min\limits_{y\in\mathbb{R}^m} & \mathbf{q}^\top y \\ \text{s.t.} & \widetilde{T}x + \widetilde{W}y \leqslant \widetilde{h} \\ & \end{bmatrix}$$

where $T \in \mathbb{R}^{p\times n}, W \in \mathbb{R}^{p\times m}$ and $h \in \mathbb{R}^p$.
We can assume $A = 0$ and $b = 0$:
We set

$$\widetilde{T} := \begin{pmatrix} T \\ A \end{pmatrix}, \quad \widetilde{W} := \begin{pmatrix} W \\ 0 \end{pmatrix} \quad \text{and } \widetilde{h} = \begin{pmatrix} h \\ b \end{pmatrix}$$

2-Stage Stochastic Linear Programming

$$\min_{x \in \mathbb{R}^n} c^\top x + V(x) \tag{2SLP}$$

where

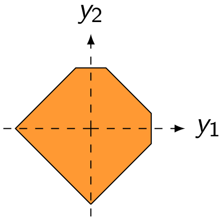
$$V(x) := \mathbb{E}\left[\begin{array}{ll} \min\limits_{y \in \mathbb{R}^m} & \mathbf{q}^\top y \\ \text{s.t.} & Tx + Wy \leqslant h \end{array}\right]$$

Fiber P_x

$$V(x) = \mathbb{E}\left[\min_{y \in P_x} \mathbf{q}^\top y\right] \quad \text{where} \quad P_x := \{y \in \mathbb{R}^m \mid Tx + Wy \leqslant h\}$$

We assume $\text{supp}(\mathbf{q}) \subset -\text{Cone}(W^\top)$ i.e. $V(x) > -\infty$. Example:

$$T = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ -1 \\ 1 \end{pmatrix} \quad W = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -1 & -1 \\ -1 & 1 \\ 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{pmatrix} \quad h = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$



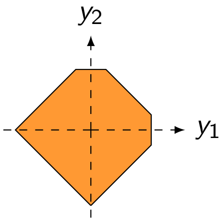
P_x for $x = 0.8$

Fiber P_x

$$V(x) = \mathbb{E}\Big[\min_{y \in P_x} \mathbf{q}^\top y\Big] \quad \text{where} \quad P_x := \{y \in \mathbb{R}^m \mid Tx + Wy \leqslant h\}$$

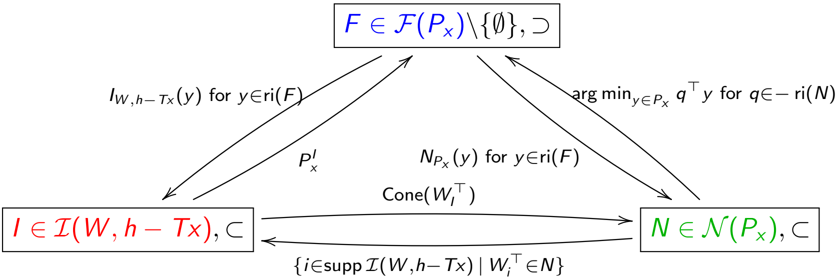
We assume $\text{supp}(\mathbf{q}) \subset -\text{Cone}(W^\top)$ i.e. $V(x) > -\infty$. Example:

$$y_1 + y_2 \leqslant 1 \qquad (1)$$
$$y_1 - y_2 \leqslant 1 \qquad (2)$$
$$-y_1 - y_2 \leqslant 1 \qquad (3)$$
$$-y_1 + y_2 \leqslant 1 \qquad (4)$$
$$y_1 \leqslant x \qquad (5)$$
$$y_2 \leqslant x \qquad (6)$$
$$x \leqslant 1.5 \qquad (7)$$



P_x for $x = 0.8$

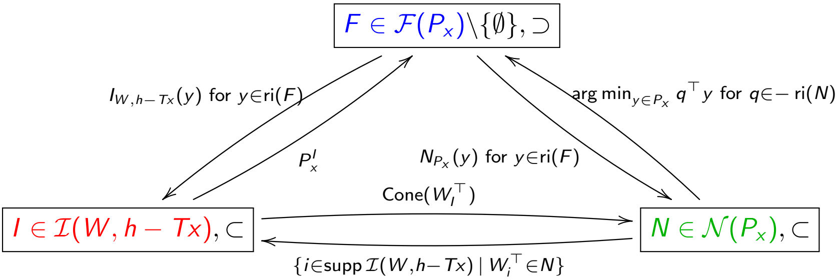
Expectation to final sum



$$V(x) = \mathbb{E}\Big[\min_{y \in P_x} \mathbf{q}^\top y\Big]$$

$$= \sum_{N \in \mathcal{N}(P_x)} \mathbb{E}\big[\mathbf{q}^\top \mathbb{1}_{\mathbf{q} \in -\mathrm{ri}\, N}\big] y_N(x) \quad \text{with } y_N(x) \in \cap_{q \in -N} \arg\min_{y \in P_x} q^\top y$$

Expectation to final sum

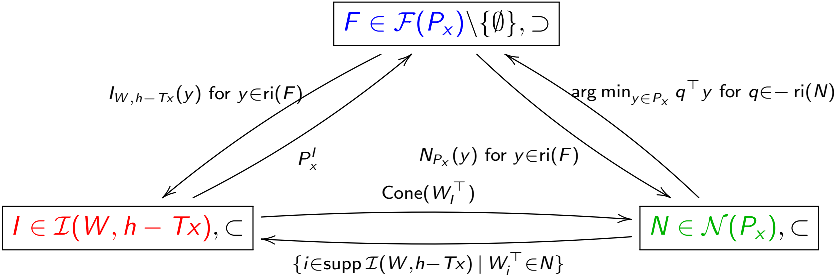


$$V(x) = \mathbb{E}\Big[\min_{y \in P_x} \mathbf{q}^\top y\Big]$$

$$= \sum_{N \in \mathcal{N}(P_x)} \mathbb{E}\big[\mathbf{q}^\top \mathbb{1}_{\mathbf{q} \in -\mathrm{ri}\, N}\big] y_N(x) \quad \text{with } y_N(x) \in \cap_{q \in -N} \arg\min_{y \in P_x} q^\top y$$

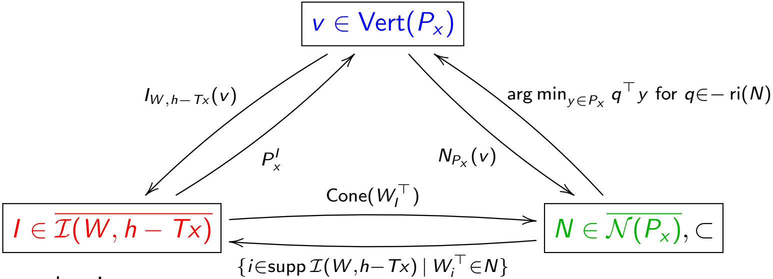
$$= \sum_{F \in \mathcal{F}(P_x)} \mathbb{E}\big[\mathbf{q}^\top \mathbb{1}_{\mathbf{q} \in -\mathrm{ri}\, N_{P_x}(F)}\big] y_F \quad \text{with } y_F \in F$$

Expectation to final sum



$$
\begin{aligned}
V(x) &= \mathbb{E}\Big[\min_{y \in P_x} \mathbf{q}^\top y\Big] \\
&= \sum_{N \in \mathcal{N}(P_x)} \mathbb{E}\Big[\mathbf{q}^\top \mathbb{1}_{\mathbf{q} \in -\mathrm{ri}\, N}\Big] y_N(x) \quad \text{with } y_N(x) \in \cap_{q \in -N} \arg\min_{y \in P_x} q^\top y \\
&= \sum_{F \in \mathcal{F}(P_x)} \mathbb{E}\Big[\mathbf{q}^\top \mathbb{1}_{\mathbf{q} \in -\mathrm{ri}\, N_{P_x}(F)}\Big] y_F \quad \text{with } y_F \in F \\
&= \sum_{I \in \mathcal{I}(W, h-Tx)} \mathbb{E}\Big[\mathbf{q}^\top \mathbb{1}_{\mathbf{q} \in -\mathrm{ri}\, \mathrm{Cone}(W_I^\top)}\Big] y_I(x) \quad \text{with } y_I(x) \in P_x^I
\end{aligned}
$$

Expectation to final sum



If $\mathbf{q}$ has a density,

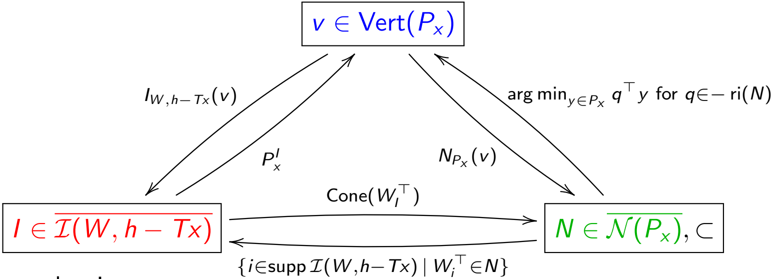
$$V(x) = \mathbb{E}\big[\min_{y \in P_x} \mathbf{q}^\top y\big]$$

$$= \sum_{N \in \overline{\mathcal{N}(P_x)}} \mathbb{E}\big[\mathbf{q}^\top \mathbb{1}_{\mathbf{q} \in -N}\big] y_N(x) \quad \text{with } y_N(x) \in \cap_{q \in -N} \arg\min_{y \in P_x} q^\top y$$

$$= \sum_{v \in \mathrm{Vert}(P_x)} \mathbb{E}\big[\mathbf{q}^\top \mathbb{1}_{\mathbf{q} \in -N_{P_x}(F)}\big] v$$

$$= \sum_{I \in \overline{\mathcal{I}(W, h - Tx)}} \mathbb{E}\big[\mathbf{q}^\top \mathbb{1}_{\mathbf{q} \in -\mathrm{Cone}(W_I^\top)}\big] y_I(x) \quad \text{with } y_I(x) \in P_x^I$$

Expectation to final sum



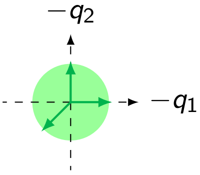
If $\mathbf{q}$ has a density,

$$
\begin{aligned}
V(x) &= \mathbb{E}\big[\min_{y \in P_x} \mathbf{q}^\top y\big] \\
&= \sum_{N \in \overline{\mathcal{N}(P_x)}} \mathbb{E}\big[\mathbf{q}^\top \mathbb{1}_{\mathbf{q} \in -N}\big] y_N(x) \quad \text{with } y_N(x) \in \cap_{q \in -N} \arg\min_{y \in P_x} q^\top y \\
&= \sum_{v \in \mathrm{Vert}(P_x)} \mathbb{E}\big[\mathbf{q}^\top \mathbb{1}_{\mathbf{q} \in -N_{P_x}(F)}\big] v \\
&= \sum_{I \in \overline{\mathcal{I}(W, h - Tx)}} \mathbb{E}\big[\mathbf{q}^\top \mathbb{1}_{\mathbf{q} \in -\mathrm{Cone}(W_I^\top)}\big] W_{B_I}^{-1}(h_{B_I} - T_{B_I} x) \text{ with basis } B_I \subset I
\end{aligned}
$$

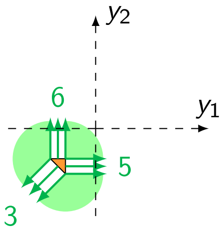
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

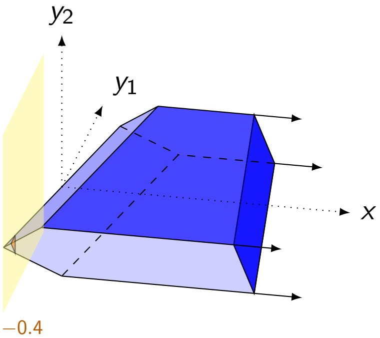
$$\text{For } x = -0.4, \quad \overline{\mathcal{I}(W, h - Tx)} = \{53, 36, 65\}$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

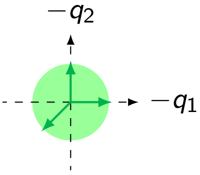


P and P_x

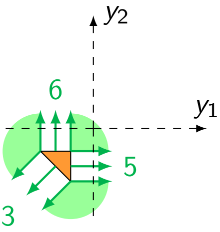
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

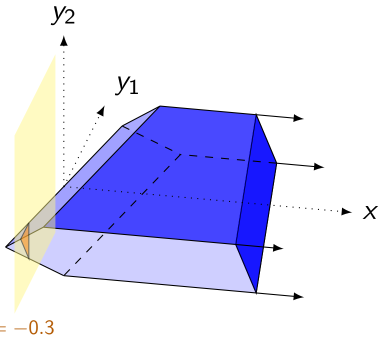
For $x = -0.3$, $\quad \overline{\mathcal{I}(W, h - Tx)} = \{53, 36, 65\}$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

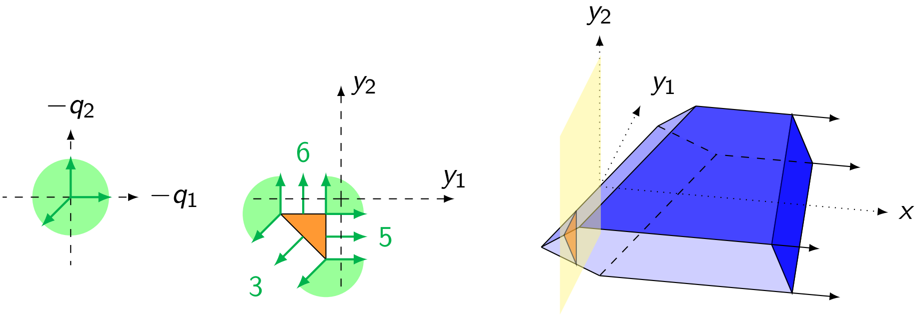


P and P_x

$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

$$\text{For } x = -0.2, \quad \overline{\mathcal{I}(W, h - Tx)} = \{53, 36, 65\}$$



$\mathcal{N}(P_x)$

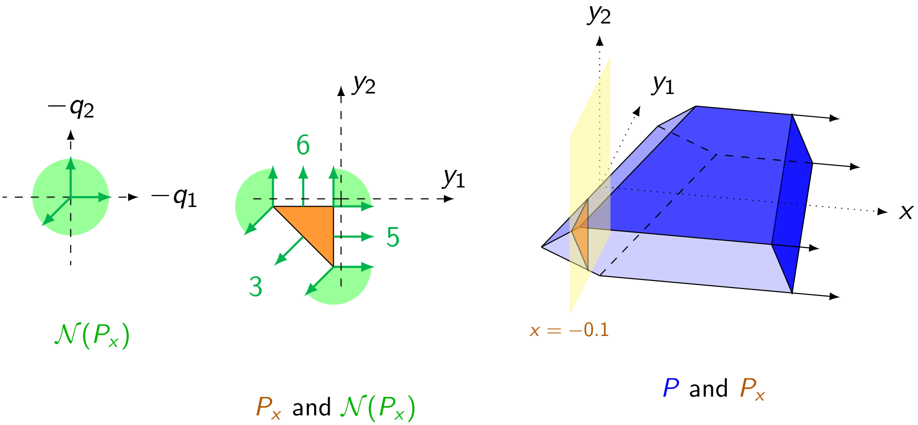
P_x and $\mathcal{N}(P_x)$

P and P_x

$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

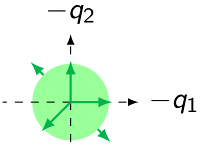
For $x = -0.1$, $\quad \overline{\mathcal{I}(W, h - Tx)} = \{53, 36, 65\}$



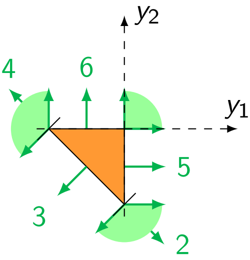
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

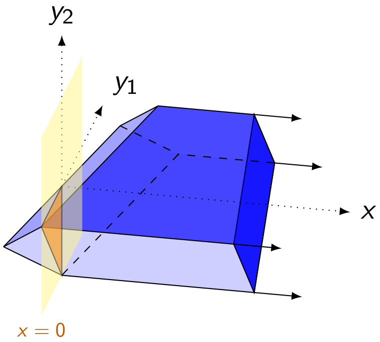
$$\text{For } x = 0, \quad \overline{\mathcal{I}(W, h - Tx)} = \{523, 346, 65\}$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

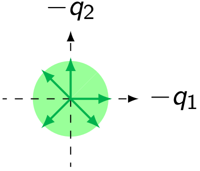


P and P_x

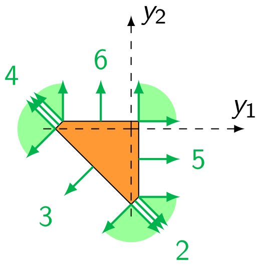
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

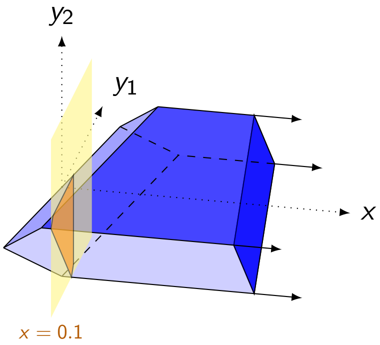
$$\text{For } x = 0.1, \quad \overline{\mathcal{I}(W, h - Tx)} = \{52, 23, 34, 46, 65\}$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

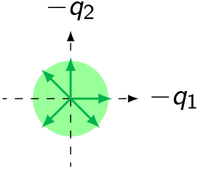


P and P_x

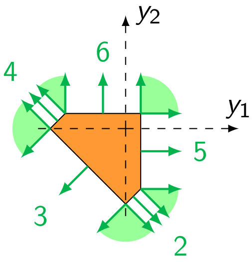
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

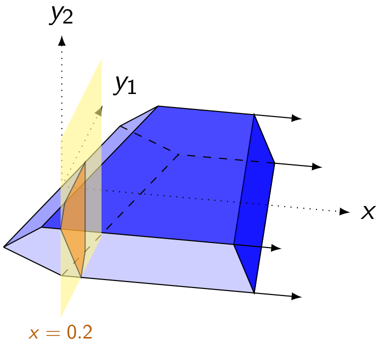
$$\text{For } x = 0.2, \quad \overline{\mathcal{I}(W, h - Tx)} = \{52, 23, 34, 46, 65\}$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

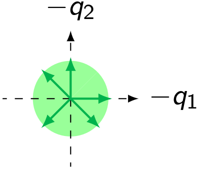


P and P_x

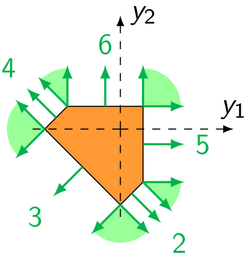
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

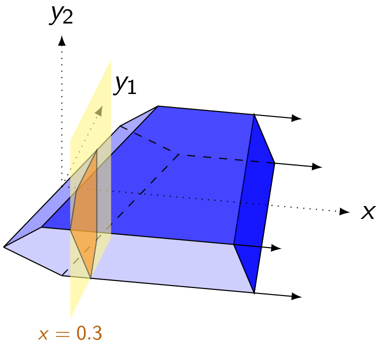
For $x = 0.3$, $\quad \overline{\mathcal{I}(W, h - Tx)} = \{52, 23, 34, 46, 65\}$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

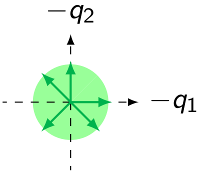


P and P_x

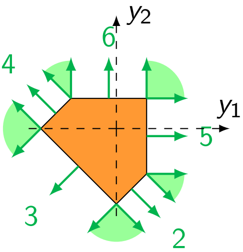
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

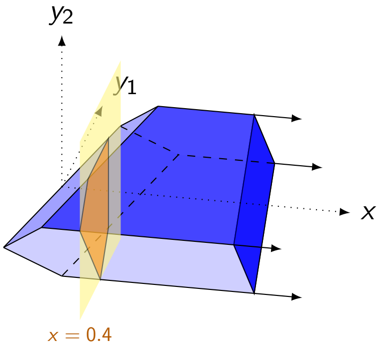
$$\text{For } x = 0.4, \quad \overline{\mathcal{I}(W, h - Tx)} = \{52, 23, 34, 46, 65\}$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

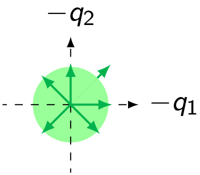


P and P_x

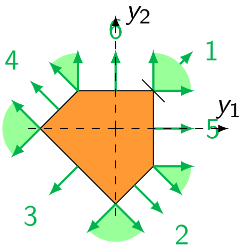
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

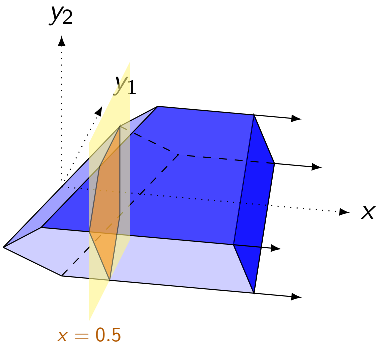
$$\text{For } x = 0.5, \quad \overline{\mathcal{I}(W, h - Tx)} = \{52, 23, 34, 46, 615\}$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

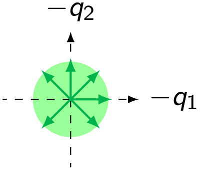


P and P_x

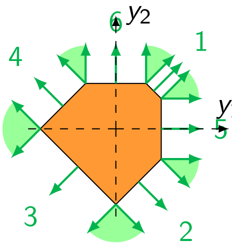
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

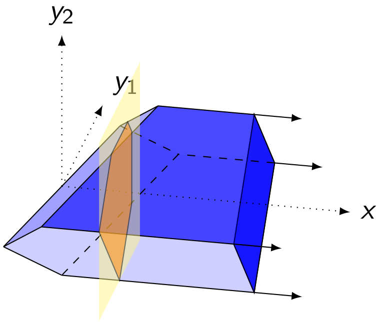
$$\text{For } x = 0.6, \quad \overline{\mathcal{I}(W, h - Tx)} = \{52, 23, 34, 46, 61, 15\}$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

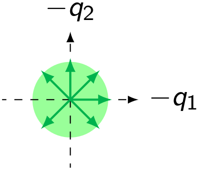


P and P_x

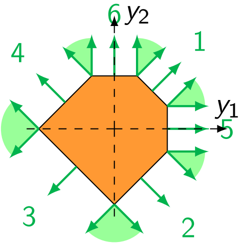
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

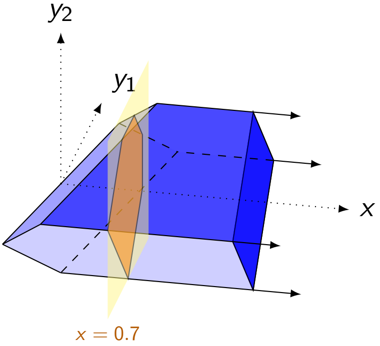
$$\text{For } x = 0.7, \quad \overline{\mathcal{I}(W, h - Tx)} = \{52, 23, 34, 46, 61, 15\}$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

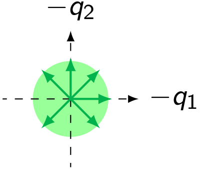


P and P_x

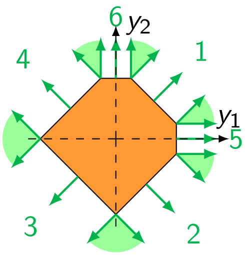
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

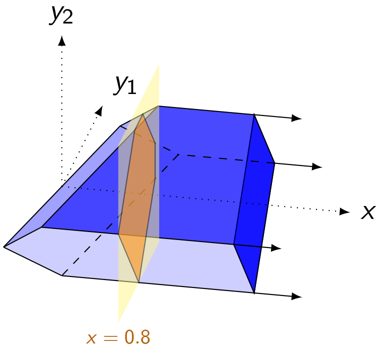
$$\text{For } x = 0.8, \quad \overline{\mathcal{I}(W, h - Tx)} = \{52, 23, 34, 46, 61, 15\}$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

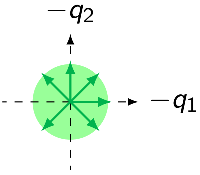


P and P_x

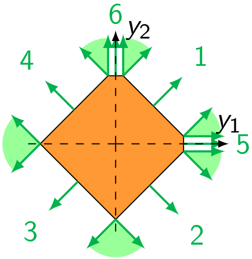
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

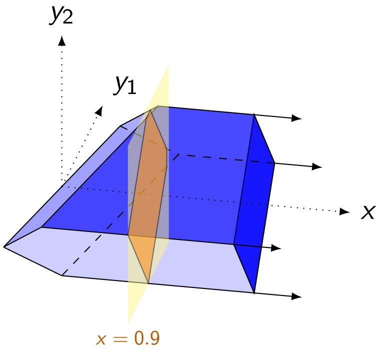
$$\text{For } x = 0.9, \quad \overline{\mathcal{I}(W, h - Tx)} = \{52, 23, 34, 46, 61, 15\}$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

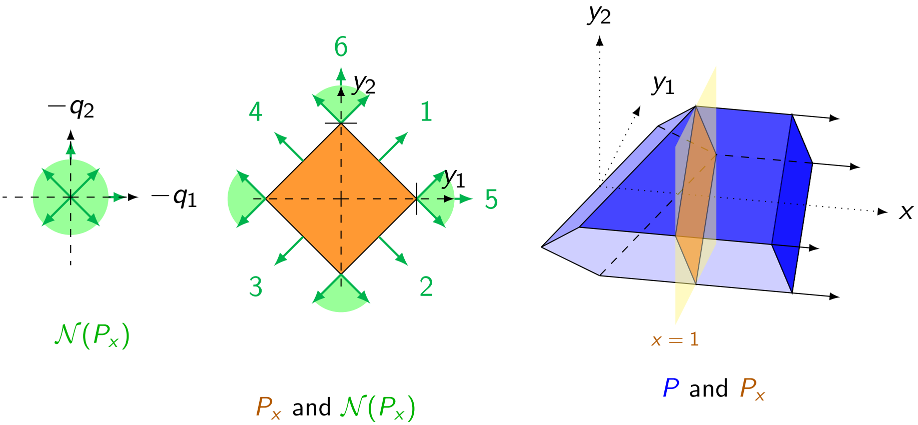


P and P_x

$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

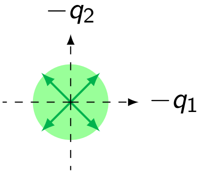
For $x = 1$, $\quad \overline{\mathcal{I}(W, h - Tx)} = \{152, 23, 34, 461\}$



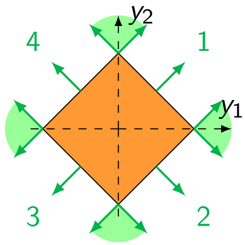
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

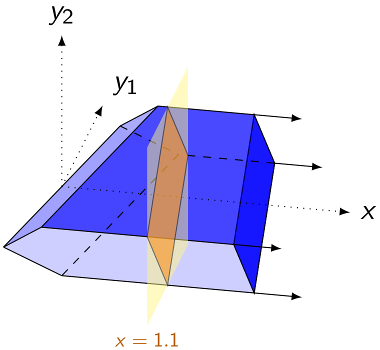
$$\text{For } x = 1.1, \quad \overline{\mathcal{I}(W, h - Tx)} = \{12, 23, 34, 41\}$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

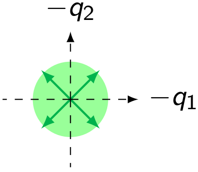


P and P_x

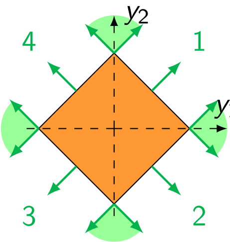
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

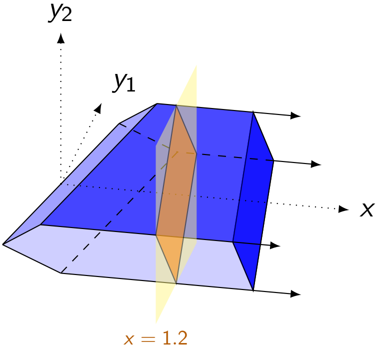
$$\text{For } x = 1.2, \quad \overline{\mathcal{I}(W, h - Tx)} = \{12, 23, 34, 41\}$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

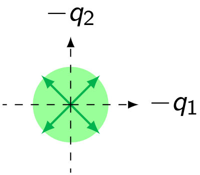


P and P_x

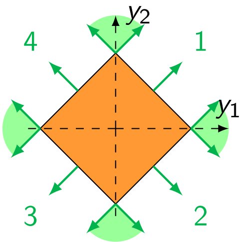
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

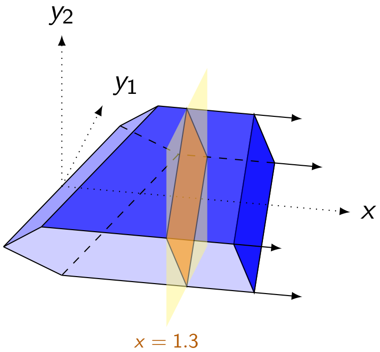
$$\text{For } x = 1.3, \quad \overline{\mathcal{I}(W, h - Tx)} = \{12, 23, 34, 41\}$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

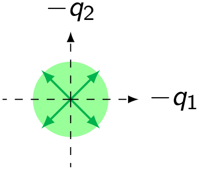


P and P_x

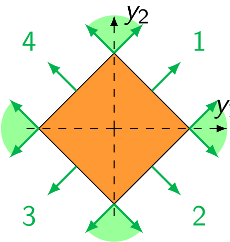
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

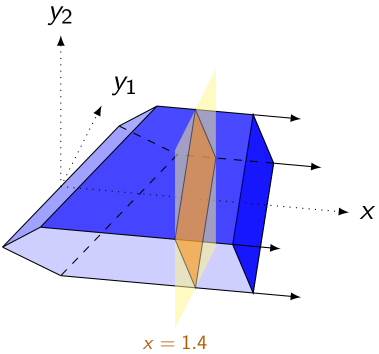
For $x = 1.4$, $\quad \overline{\mathcal{I}(W, h - Tx)} = \{12, 23, 34, 41\}$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

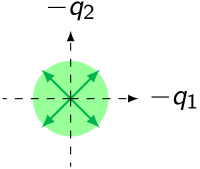


P and P_x

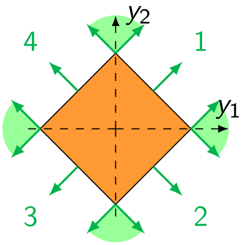
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

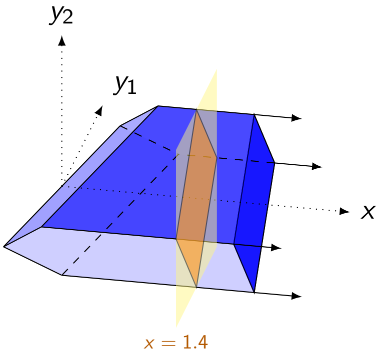
For $x = 1.4$, $\quad \overline{\mathcal{I}(W, h - Tx)} = \{12, 23, 34, 41\}$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

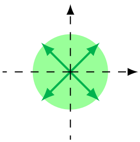


P and P_x

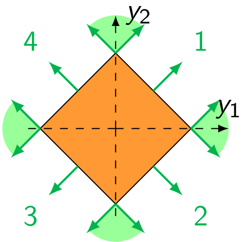
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

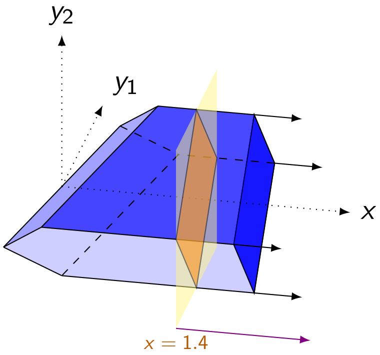
For $x = 1.4$, $\quad \overline{\mathcal{I}(W, h - Tx)} = \{12, 23, 34, 41\}$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

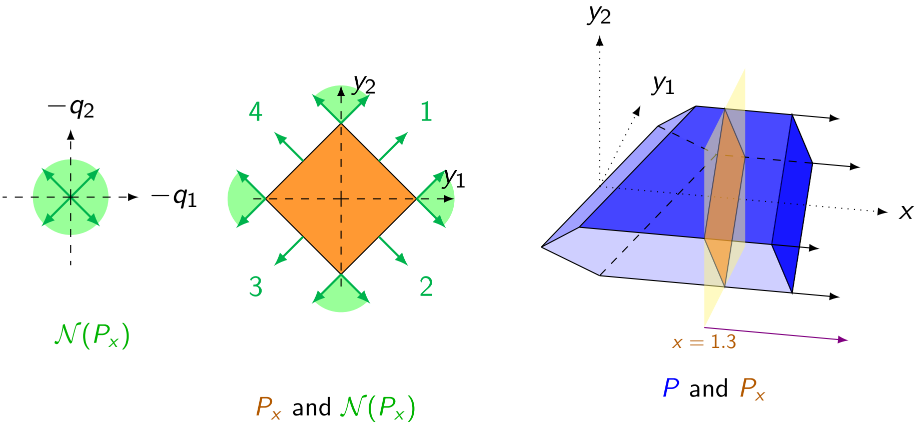


P and P_x

$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

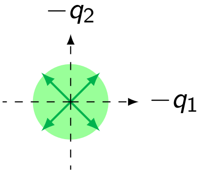
For $x = 1.3$, $\quad \overline{\mathcal{I}(W, h - Tx)} = \{12, 23, 34, 41\}$



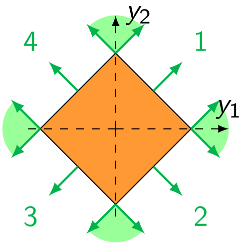
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

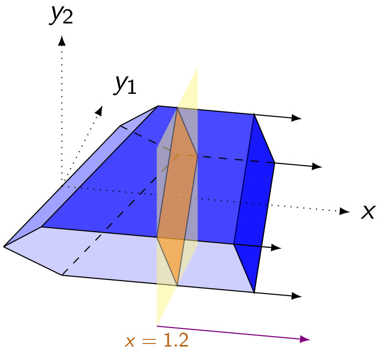
For $x = 1.2$, $\quad \overline{\mathcal{I}(W, h - Tx)} = \{12, 23, 34, 41\}$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

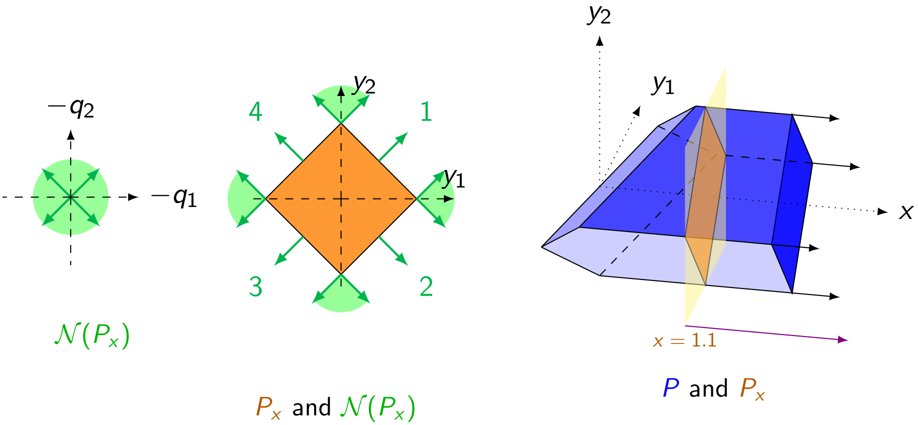


P and P_x

$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

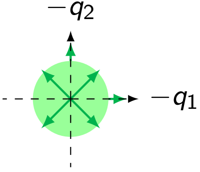
For $x = 1.1$, $\quad \overline{\mathcal{I}(W, h - Tx)} = \{12, 23, 34, 41\}$



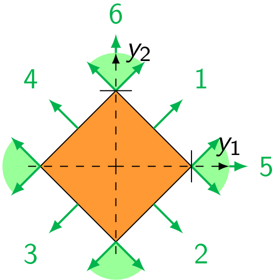
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

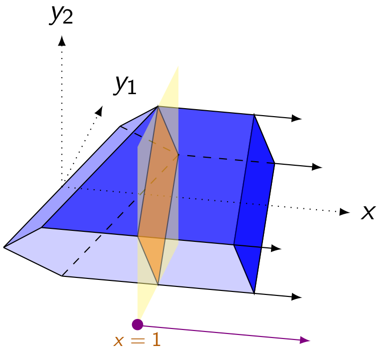
For $x = 1$, $\quad \overline{\mathcal{I}(W, h - Tx)} = \{152, 23, 34, 461\}$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

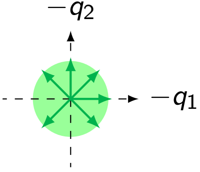


P and P_x

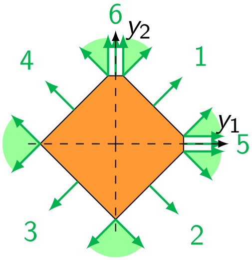
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

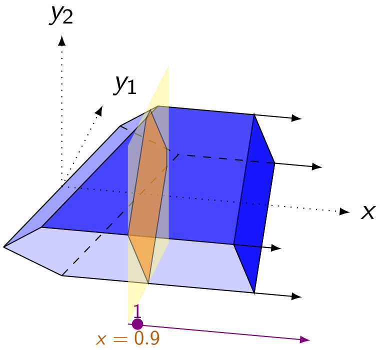
For $x = 0.9$, $\quad \overline{\mathcal{I}(W, h - Tx)} = \{52, 23, 34, 46, 61, 15\}$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

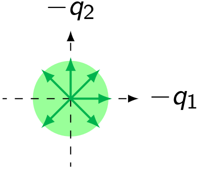


P and P_x

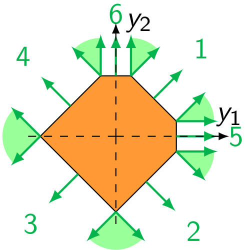
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

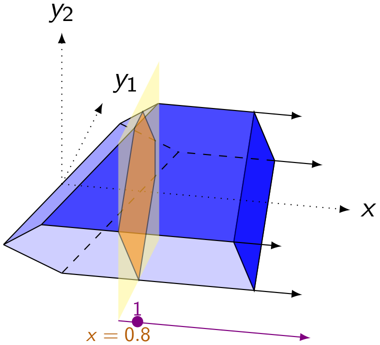
$$\text{For } x = 0.8, \quad \overline{\mathcal{I}(W, h - Tx)} = \{52, 23, 34, 46, 61, 15\}$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

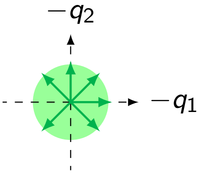


P and P_x

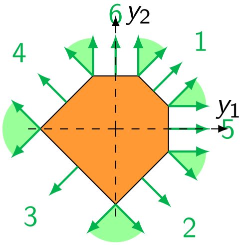
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

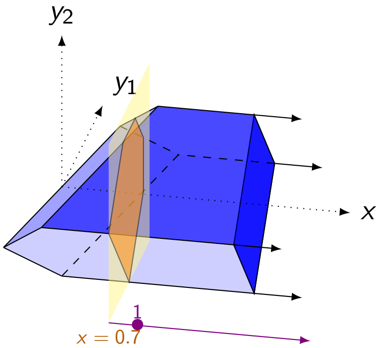
$$\text{For } x = 0.7, \quad \overline{\mathcal{I}(W, h - Tx)} = \{52, 23, 34, 46, 61, 15\}$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

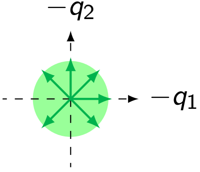


P and P_x

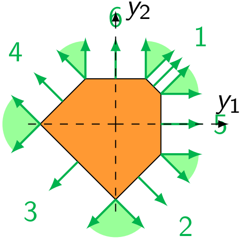
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

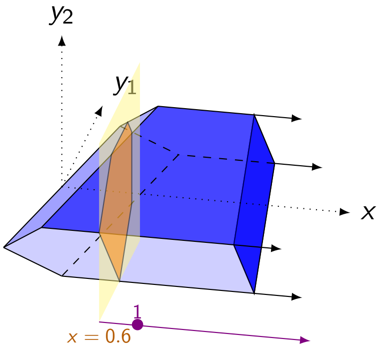
For $x = 0.6$, $\quad \overline{\mathcal{I}(W, h - Tx)} = \{52, 23, 34, 46, 61, 15\}$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

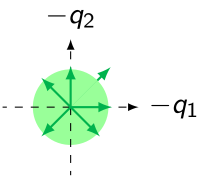


P and P_x

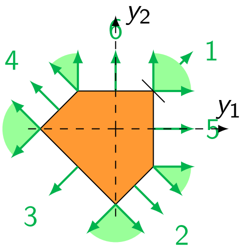
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

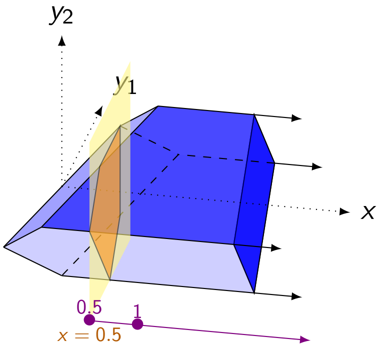
$$\text{For } x = 0.5, \quad \overline{\mathcal{I}(W, h - Tx)} = \{52, 23, 34, 46, 615\}$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

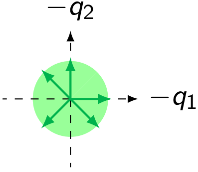


P and P_x

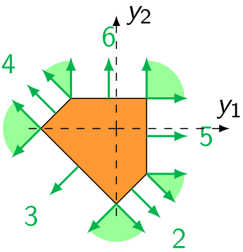
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

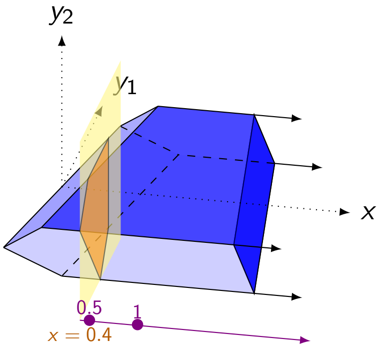
For $x = 0.4$, $\quad \overline{\mathcal{I}(W, h - Tx)} = \{52, 23, 34, 46, 65\}$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

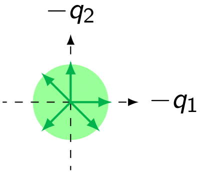


P and P_x

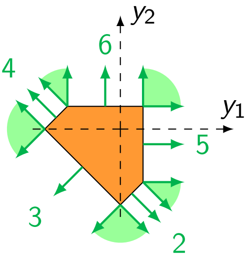
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

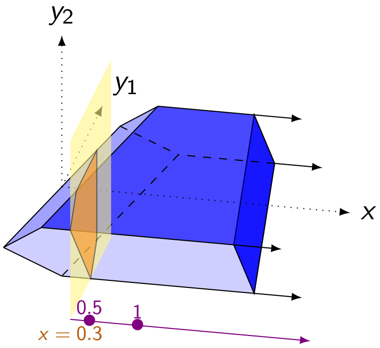
For $x = 0.3$, $\quad \overline{\mathcal{I}(W, h - Tx)} = \{52, 23, 34, 46, 65\}$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

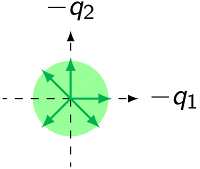


P and P_x

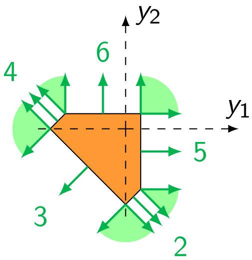
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

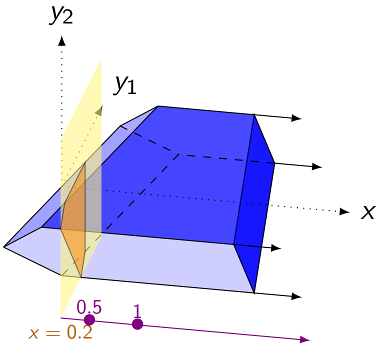
$$\text{For } x = 0.2, \quad \overline{\mathcal{I}(W, h - Tx)} = \{52, 23, 34, 46, 65\}$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

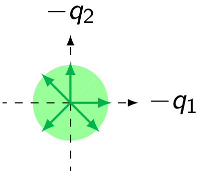


P and P_x

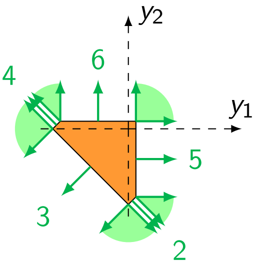
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

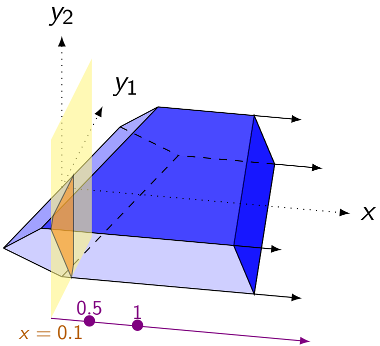
For $x = 0.1$, $\quad \overline{\mathcal{I}(W, h - Tx)} = \{52, 23, 34, 46, 65\}$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

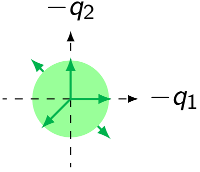


P and P_x

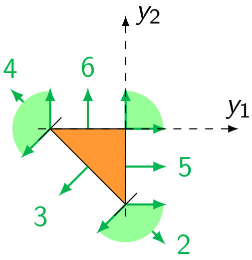
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

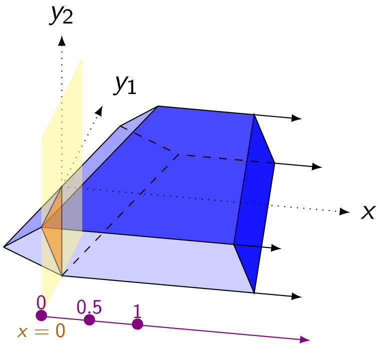
$$\text{For } x = 0, \quad \overline{\mathcal{I}(W, h - Tx)} = \{523, 346, 65\}$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

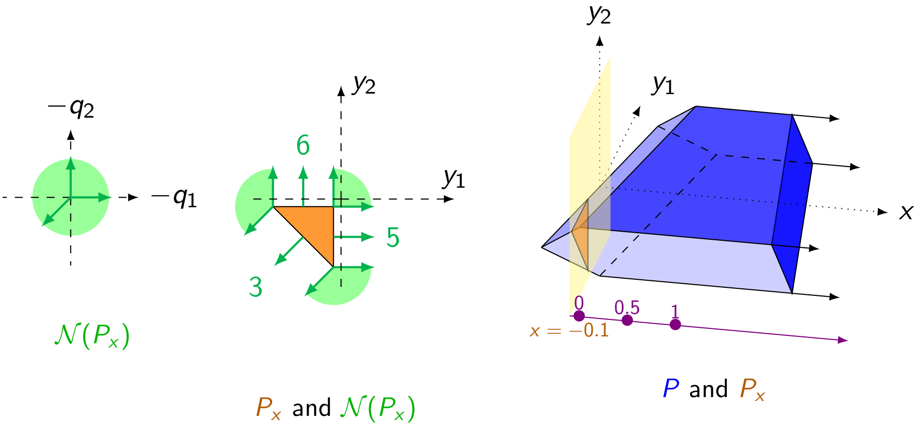


P and P_x

$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

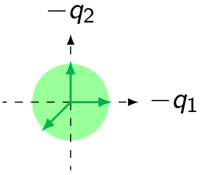
$$\text{For } x = -0.1, \quad \overline{\mathcal{I}(W, h - Tx)} = \{53, 36, 65\}$$



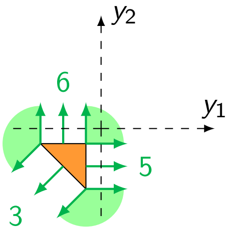
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

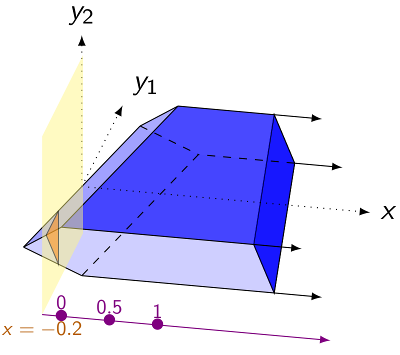
For $x = -0.2$, $\quad \overline{\mathcal{I}(W, h - Tx)} = \{53, 36, 65\}$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

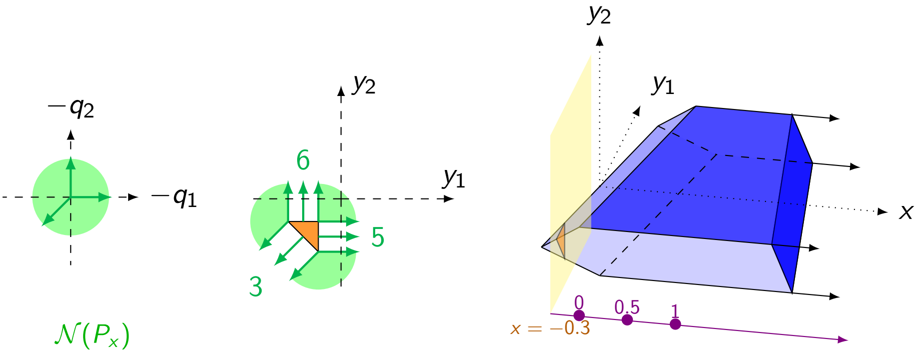


P and P_x

$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

For $x = -0.3$, $\quad \overline{\mathcal{I}(W, h - Tx)} = \{53, 36, 65\}$

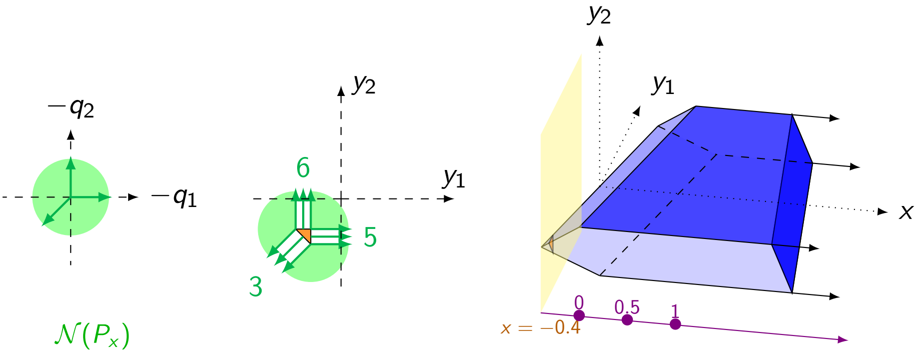


$\mathcal{N}(P_x)$ $\quad$ P_x and $\mathcal{N}(P_x)$ $\quad$ P and P_x

$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

For $x = -0.4$, $\quad \overline{\mathcal{I}(W, h - Tx)} = \{53, 36, 65\}$



$\mathcal{N}(P_x)$

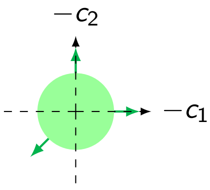
P_x and $\mathcal{N}(P_x)$

P and P_x

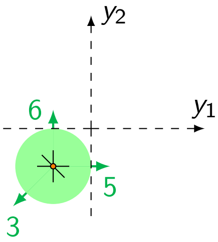
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

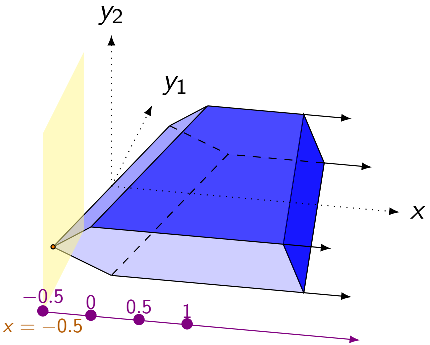
$$\text{For } x = -0.5, \quad \overline{\mathcal{I}(W, h - Tx)} = \{536\}$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$



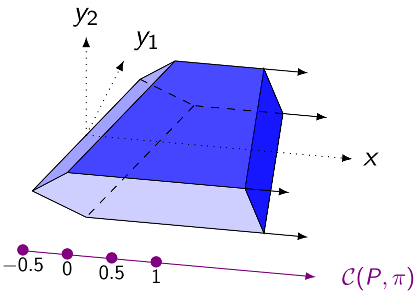
P and P_x

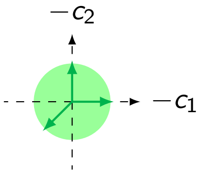
What are the constant regions of $\mathcal{N}(P_x)$, $\mathcal{I}(W, h - Tx)$?

Lemma

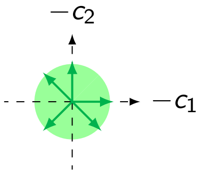
There exists a collection $\mathcal{C}(P,\pi)$ whose relative interior of cells are the constant regions of $x \to \mathcal{N}(P_x)$ and $x \to \mathcal{I}(W, h - Tx)$.

For $\sigma \in \mathcal{C}(P,\pi)$ and $x, x' \in \mathrm{ri}(\sigma)$,
$\mathcal{N}(P_x) = \mathcal{N}(P_{x'}) = \mathcal{N}_\sigma$
$\mathcal{I}(W, h - Tx) = \mathcal{I}(W, h - Tx') = \mathcal{I}_\sigma$

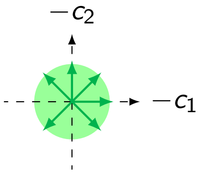




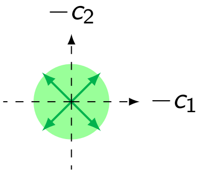
$\mathcal{N}_\sigma$ for $\sigma = [-0.5, 0]$



$\mathcal{N}_\sigma$ for $\sigma = [0, 0.5]$



$\mathcal{N}_\sigma$ for $\sigma = [0.5, 1]$



$\mathcal{N}_\sigma$ for $\sigma = [1, +\infty)$

Chamber complex

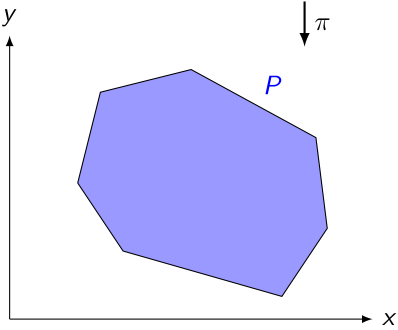
Definition

The *chamber complex* $\mathcal{C}(P, \pi)$ of P along π is

$$\mathcal{C}(P, \pi) := \{\sigma_{P,\pi}(x) \mid x \in \pi(P)\}$$

where

$$\sigma_{P,\pi}(x) := \bigcap_{F \in \mathcal{F}(P) \text{ s.t. } x \in \pi(F)} \pi(F)$$



where $\mathcal{F}(P)$ is the set of faces of P and π is the projection $(x, y) \to x$

$$\pi(E) := \{x \in \mathbb{R}^n \mid \exists y \in \mathbb{R}^m,\ (x, y) \in E\}$$

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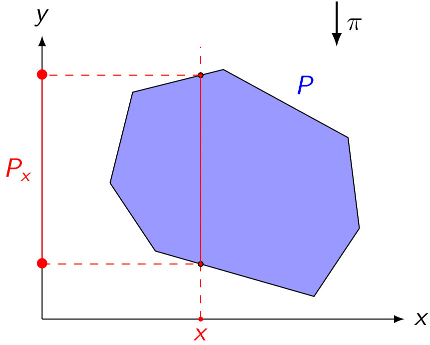

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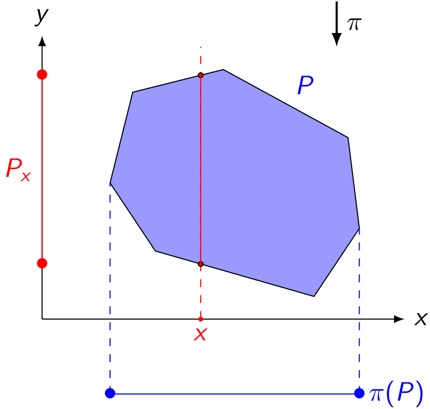

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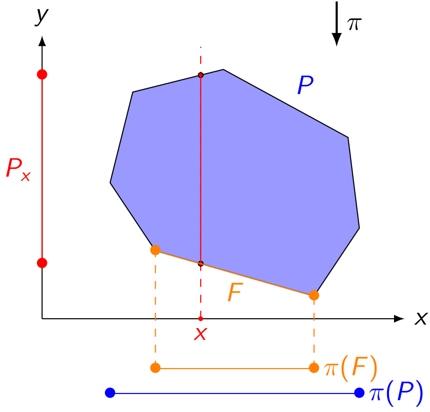

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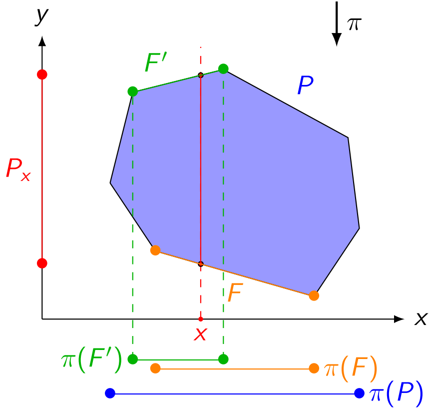

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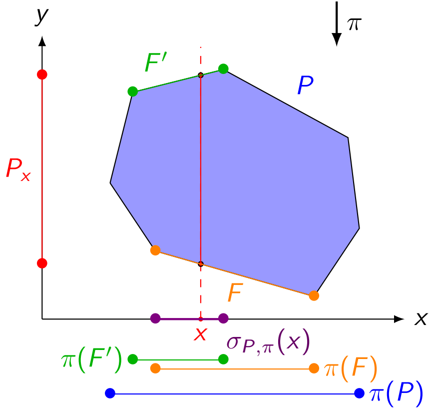

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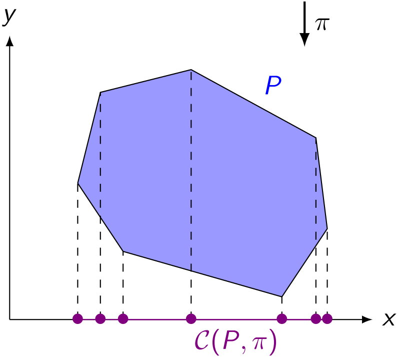

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H-representation of projection of faces

Let $I \in \mathcal{I}\big((T,W),h\big)$ be a set of indices

$$x \in \pi(P^I) \iff \begin{cases} \exists y \in \mathbb{R}^m, & (x,y) \in P^I \\ \end{cases}$$

H-representation of projection of faces

Let $I \in \mathcal{I}\big((T,W),h\big)$ be a set of indices

$$x \in \pi(P^I) \iff \begin{cases} \exists y \in \mathbb{R}^m, & T_I x + W_I y = h_I \\ \quad \forall j \in [q] \backslash I, & T_j x + W_j y \leqslant h_j \end{cases} \Longleftarrow I \in \mathcal{I}(W, h - Tx)$$

H-representation of projection of faces

Let $I \in \mathcal{I}\big((T, W), h\big)$ be a set of indices

$$x \in \operatorname{ri} \pi(P^I) \iff \begin{cases} \exists y \in \mathbb{R}^m, & T_I x + W_I y = h_I \\ \quad \forall j \in [q] \backslash I, & T_j x + W_j y < h_j \end{cases} \iff I \in \mathcal{I}(W, h - Tx)$$

H-representation of projection of faces

Let $I \in \mathcal{I}\big((T, W), h\big)$ be a set of indices from which we can extract a basis (i.e. $\mathrm{rg}(W_I^\top) = m$) and let B such a basis

$$x \in \operatorname{ri} \pi(P^I) \iff \begin{cases} \exists y \in \mathbb{R}^m, & T_B x + W_B y = h_B \\ \forall i \in I \backslash B, & T_i x + W_i y = h_i \\ \forall j \in [q] \backslash I, & T_j x + W_j y < h_j \end{cases} \iff I \in \mathcal{I}(W, h - Tx)$$

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$$x \in \operatorname{ri} \pi(P^I) \iff \begin{cases} \exists y \in \mathbb{R}^m, & y = W_B^{-1}(h_B - T_B x) \\ \forall i \in I \backslash B, & T_i x + W_i y = h_i \\ \forall j \in [q] \backslash I, & T_j x + W_j y < h_j \end{cases} \iff I \in \mathcal{I}(W, h - Tx)$$

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$$x \in \mathrm{ri}\,\pi(P^I) \iff \begin{cases} \forall i \in I \backslash B, \quad T_i x + W_i W_B^{-1}(h_B - T_B x) = h_i \\ \forall j \in [q] \backslash I, \quad T_j x + W_j W_B^{-1}(h_B - T_B x) < h_j \end{cases}$$

H-representation of projection of faces

Let $I \in \mathcal{I}\big((T, W), h\big)$ be a set of indices from which we can extract a basis (i.e. $\mathrm{rg}(W_I^\top) = m$) and let B such a basis

$$x \in \mathrm{ri}\left(\pi(P^I)\right) \iff \begin{cases} \forall i \in I \backslash B, & (v_i^B)^\top x = u_i^B \\ \forall j \in [q] \backslash I, & (v_j^B)^\top x < u_j^B \end{cases} \iff I \in \mathcal{I}(W, h - Tx)$$

where

$$v_i^B := (T_i - W_i W_B^{-1} T_B)^\top$$
$$u_i^B := h_i - W_i W_B^{-1} h_B$$

H-representation of chambers

Let $\sigma \in \mathcal{C}(P, \pi)$

$$x \in \bigcap_{I \in \overline{\mathcal{I}_\sigma}} \operatorname{ri}\left(\pi(P^I)\right) \iff \begin{cases} \forall I \in \overline{\mathcal{I}_\sigma}, \\ \forall i \in I \backslash B_I, \quad (v_i^{B_I})^\top x = u_i^{B_I} \\ \forall j \in [q] \backslash I, \quad (v_j^{B_I})^\top x < u_j^{B_I} \end{cases} \iff \mathcal{I}(W, h - Tx) = \mathcal{I}_\sigma$$

where

$$\begin{aligned} v_i^B &:= (T_i - W_i W_B^{-1} T_B)^\top \\ u_i^B &:= h_i - W_i W_B^{-1} h_B \end{aligned}$$

with B_I basis $\subset I$ and

$$\begin{aligned} \mathcal{G}_\sigma &:= \{F \in \mathcal{F}(P) \,|\, \sigma \subset \pi(F)\} \\ \mathcal{I}_\sigma &:= \{I \in \mathcal{I}\big((T, W), h\big) \,|\, \sigma \subset \pi(P^I)\} \end{aligned}$$

We have $\sigma = \cap_{G \in \mathcal{G}_\sigma} \pi(G) = \cap_{I \in \mathcal{I}_\sigma} \pi(P^I)$

H-representation of chambers

Let $\sigma \in \mathcal{C}(P, \pi)$

$$x \in \mathrm{ri}(\sigma) \iff \begin{cases} \forall I \in \overline{\mathcal{I}_\sigma}, \\ \forall i \in I \backslash B_I, \quad (v_i^{B_I})^\top x = u_i^{B_I} \\ \forall j \in [q] \backslash I, \quad (v_j^{B_I})^\top x < u_j^{B_I} \end{cases} \iff \mathcal{I}(W, h - Tx) = \mathcal{I}_\sigma$$

where

$$v_i^B := (T_i - W_i W_B^{-1} T_B)^\top$$
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with B_I basis $\subset I$ and

$$\mathcal{G}_\sigma := \{F \in \mathcal{F}(P) \,|\, \sigma \subset \pi(F)\}$$
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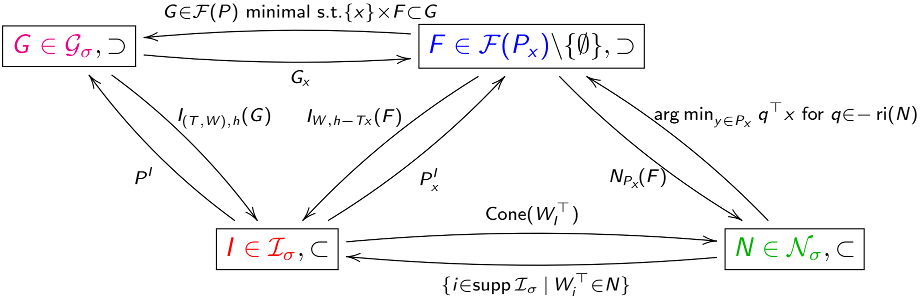
We have $\sigma = \cap_{G \in \underline{\mathcal{G}_\sigma}} \pi(G) = \cap_{I \in \overline{\mathcal{I}_\sigma}} \pi(P^I)$

Correspondences



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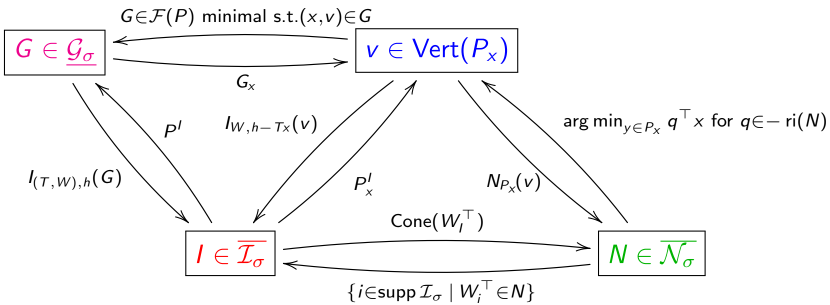

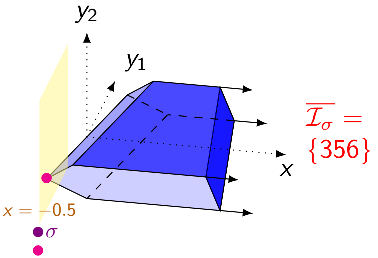

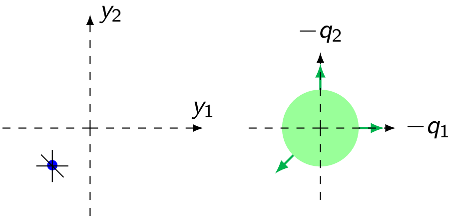

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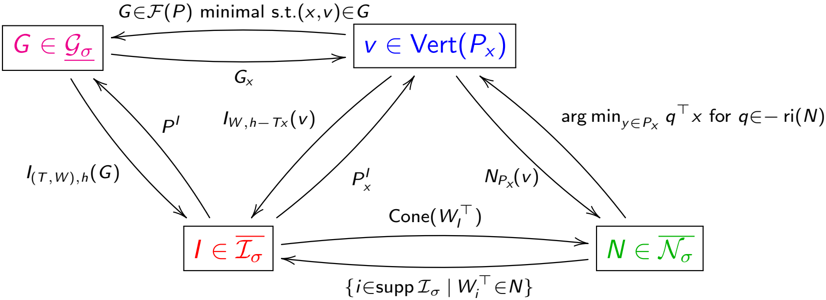
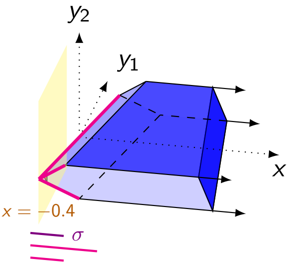
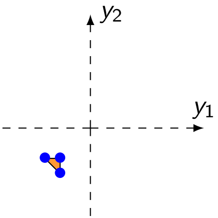
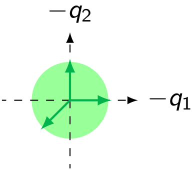




$\overline{\mathcal{I}_\sigma} = \{53, 36, 65\}$





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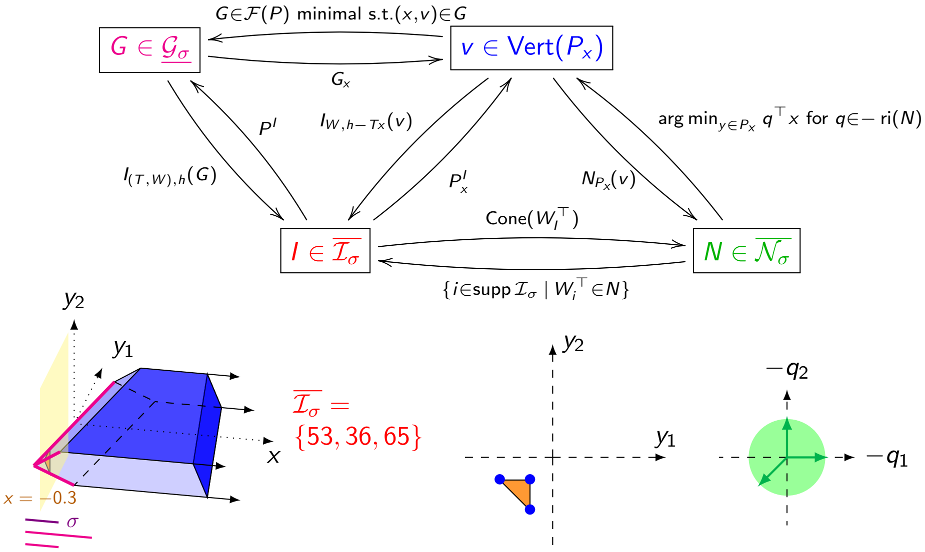

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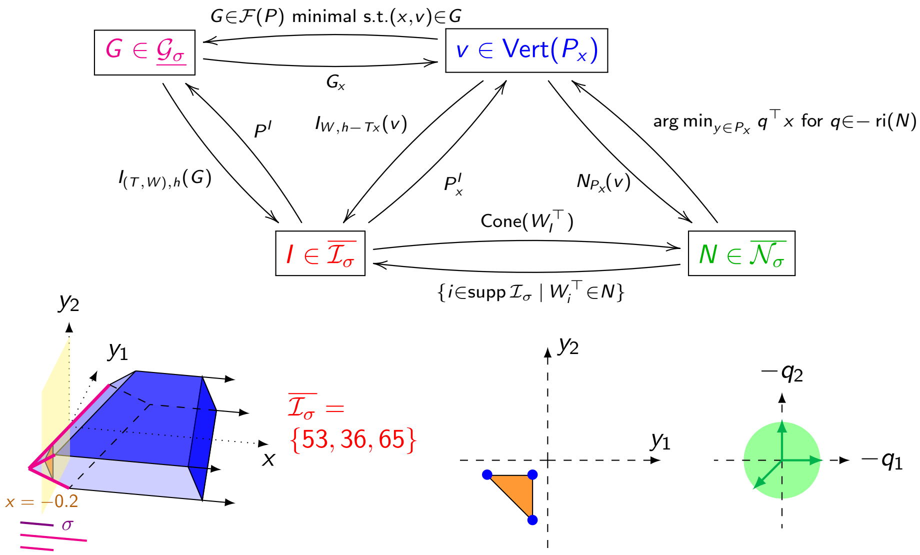

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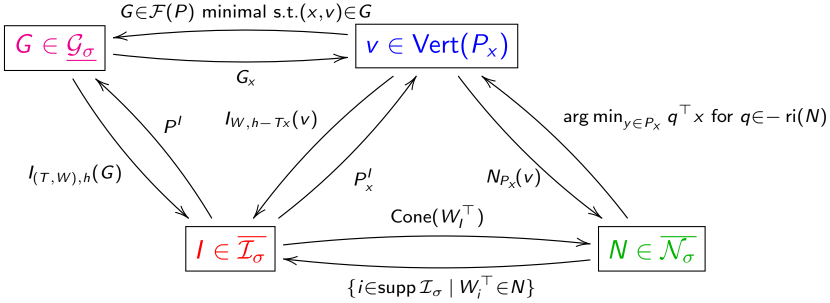

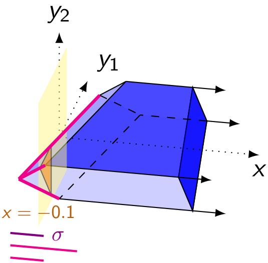

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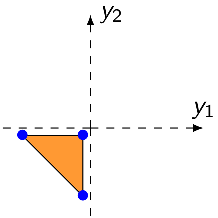

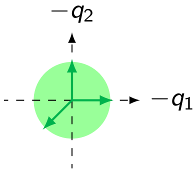

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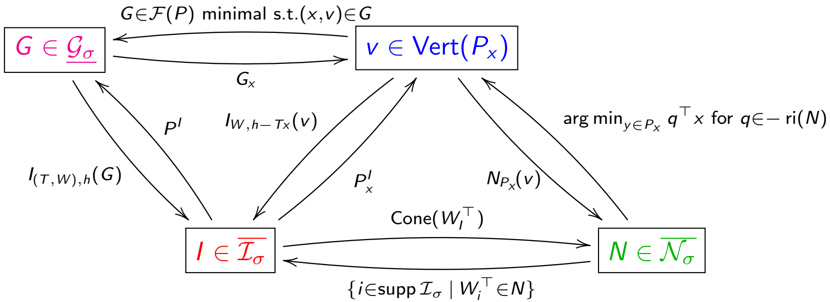
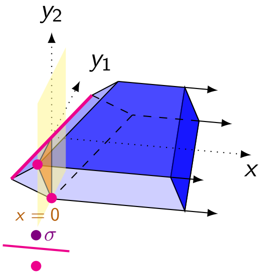
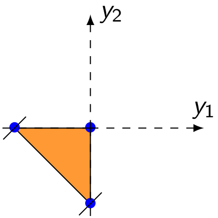
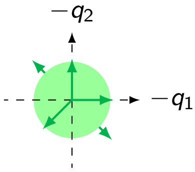


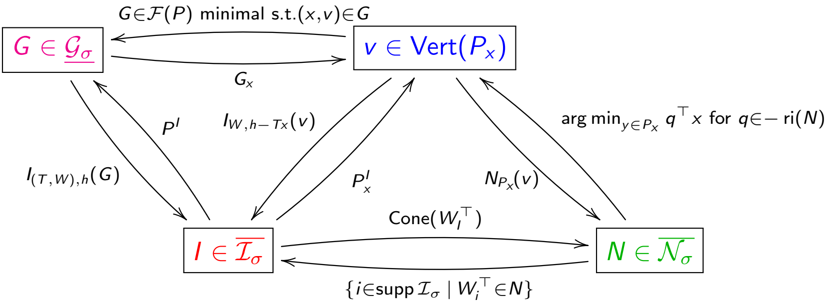


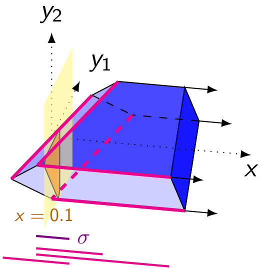
$\overline{\mathcal{I}_\sigma} =$
$\{523, 346, 65\}$



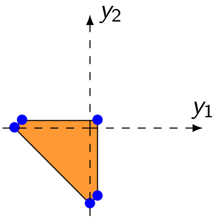


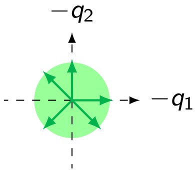
Correspondences





$\overline{\mathcal{I}_\sigma} = \{52, 23, 34, 46, 65\}$





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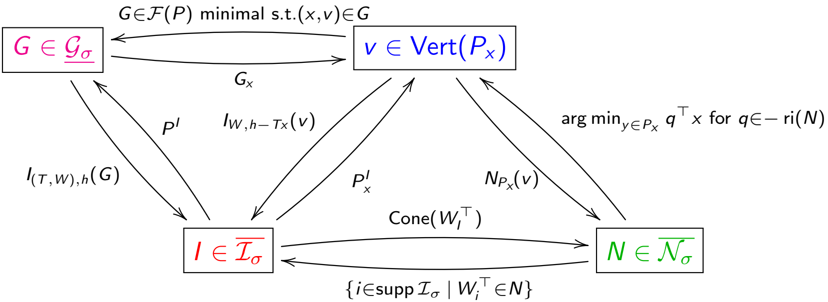
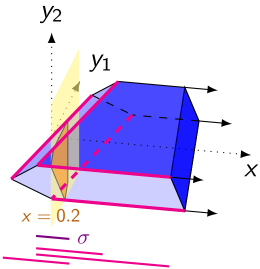
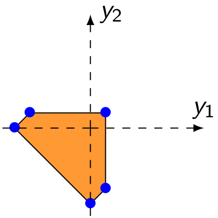
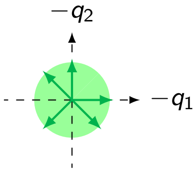




$\overline{\mathcal{I}_\sigma} = \{52, 23, 34, 46, 65\}$





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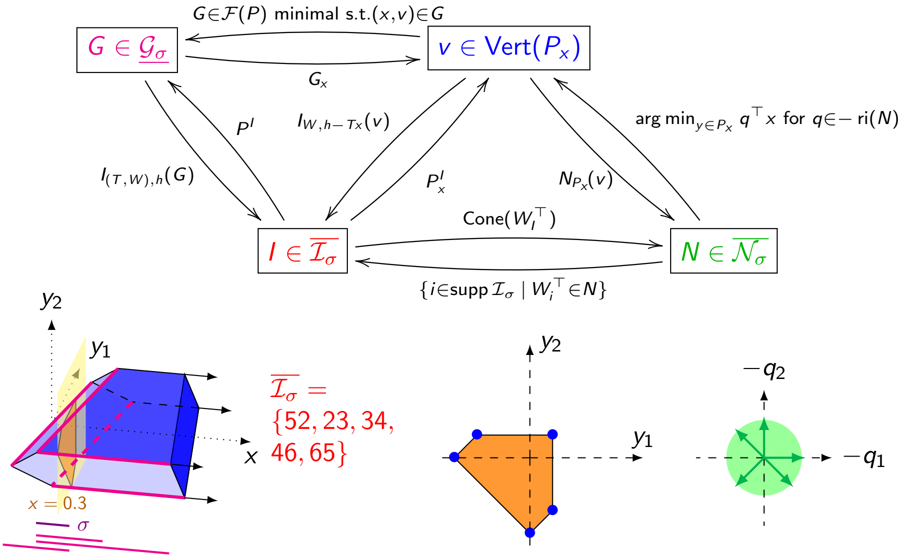

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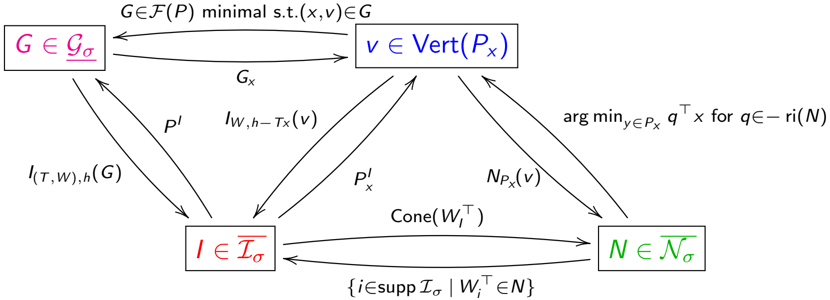
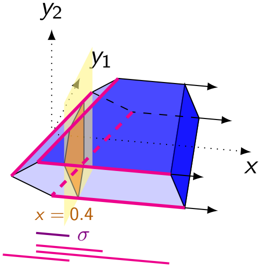
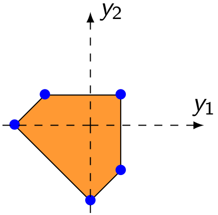
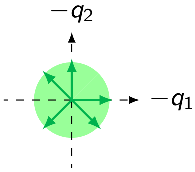




$\overline{\mathcal{I}_\sigma} = \{52, 23, 34, 46, 65\}$





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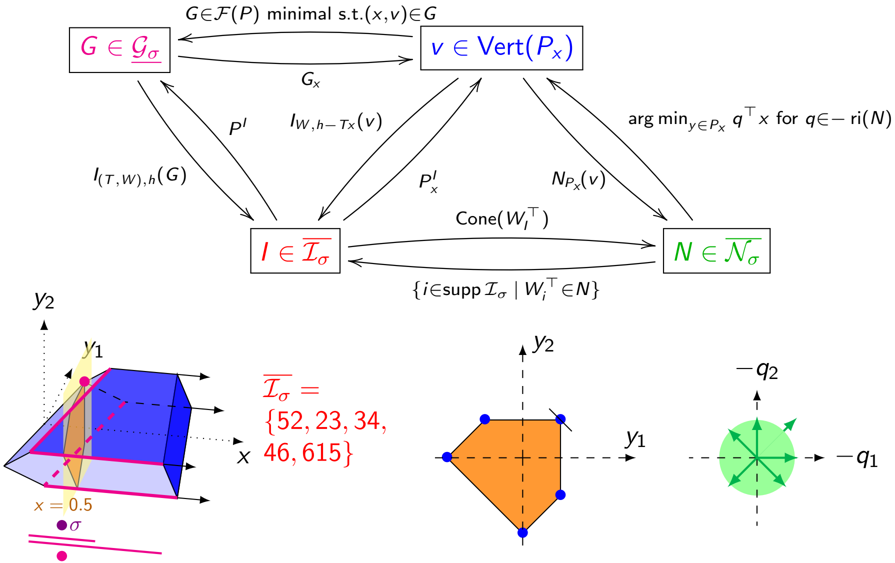

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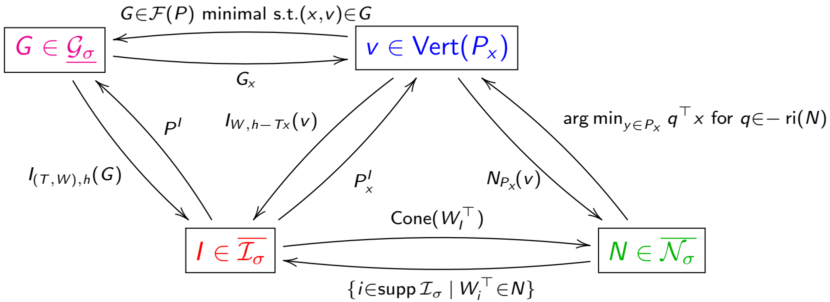
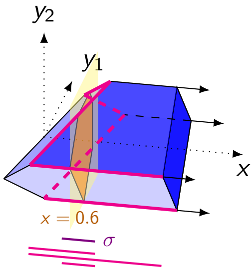
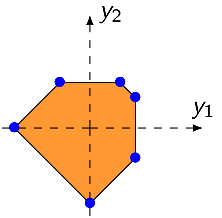
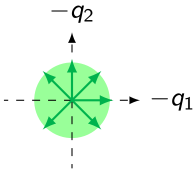




$\overline{\mathcal{I}_\sigma} = \{52, 23, 34, 46, 61, 15\}$





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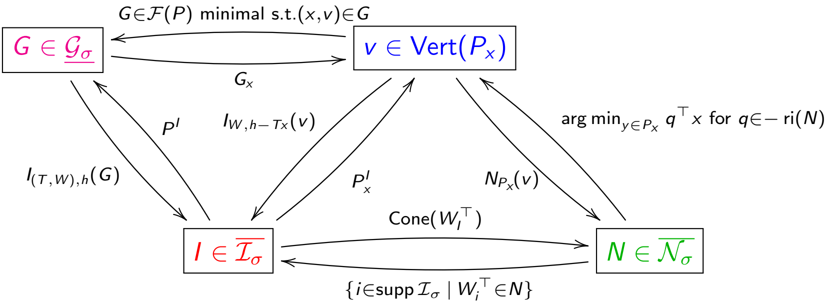
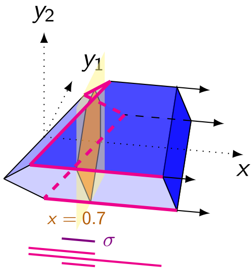
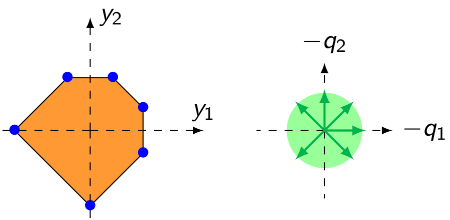




$\overline{\mathcal{I}_\sigma} = \{52, 23, 34, 46, 61, 15\}$



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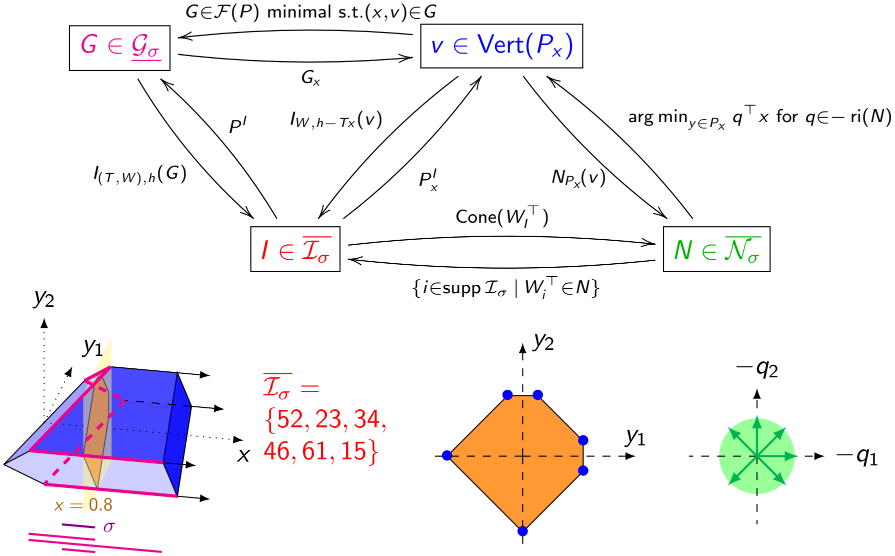

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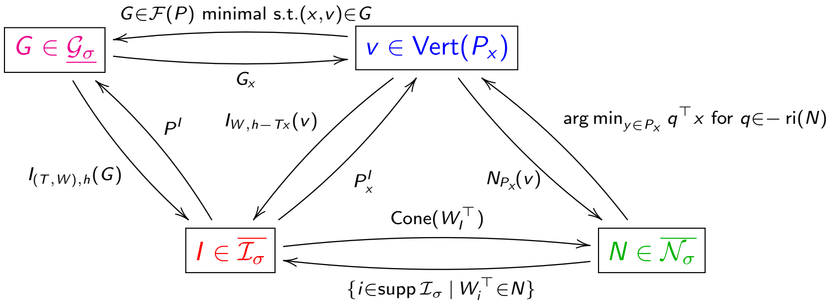
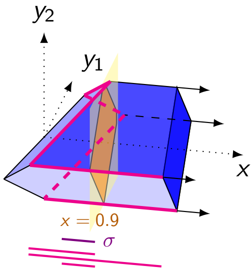
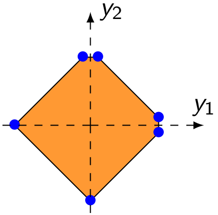
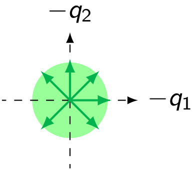




$\overline{\mathcal{I}_\sigma} = \{52, 23, 34, 46, 61, 15\}$





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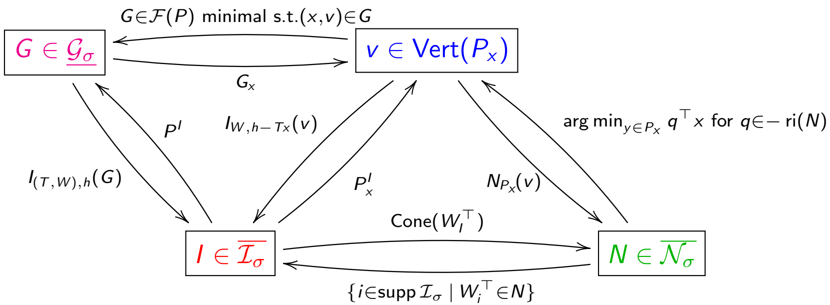
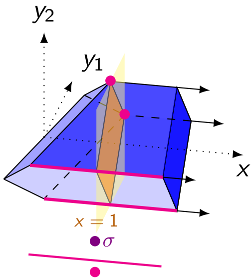
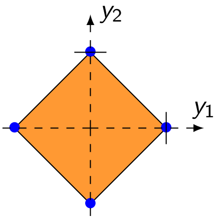
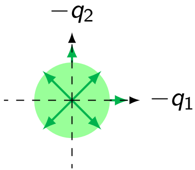




$\overline{\mathcal{I}_\sigma} = \{152, 23, 34, 461\}$





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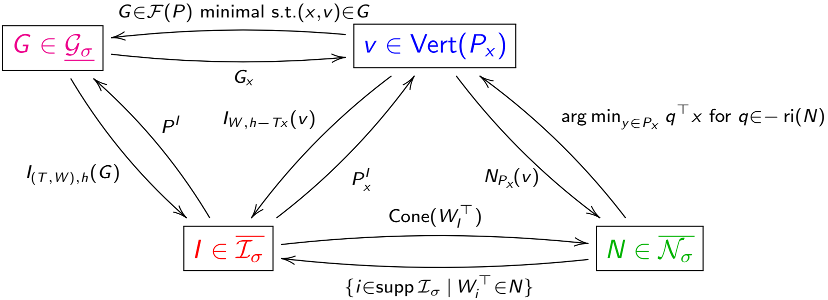
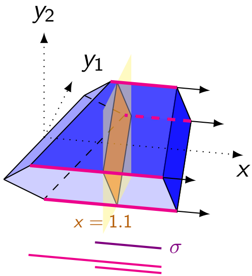
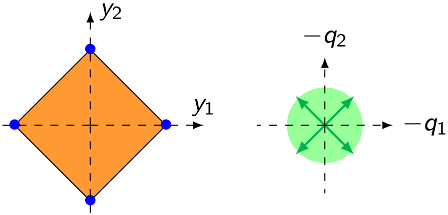




$\overline{\mathcal{I}_\sigma} = \{12, 23, 34, 41\}$



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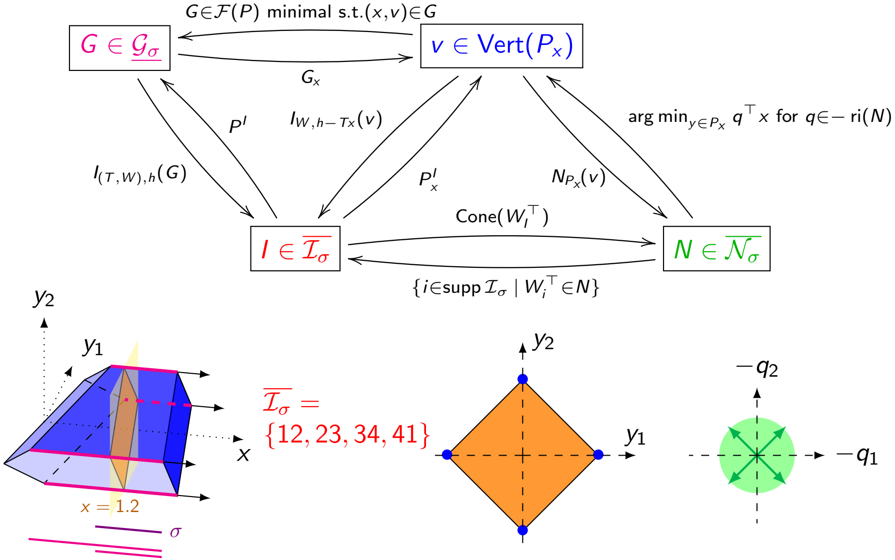

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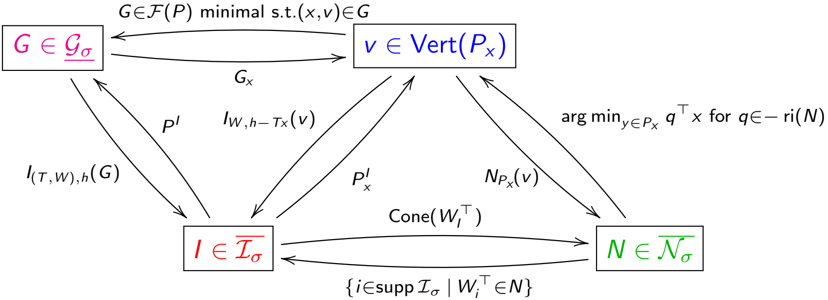
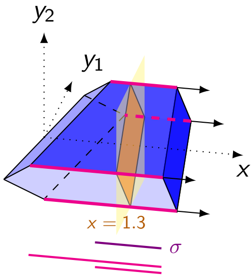
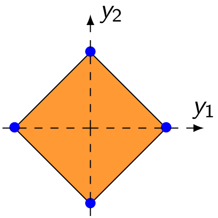
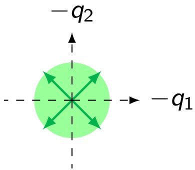




$\overline{\mathcal{I}_\sigma} =$
$\{12, 23, 34, 41\}$





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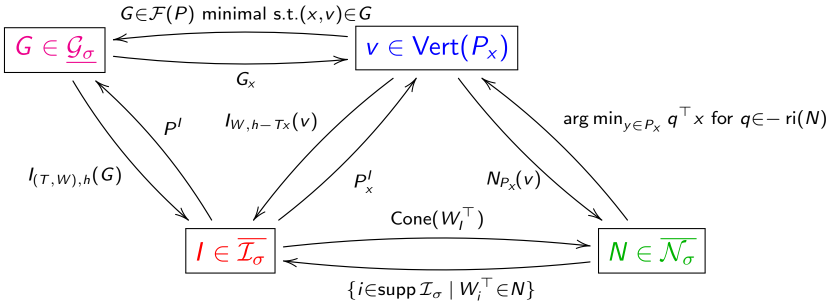
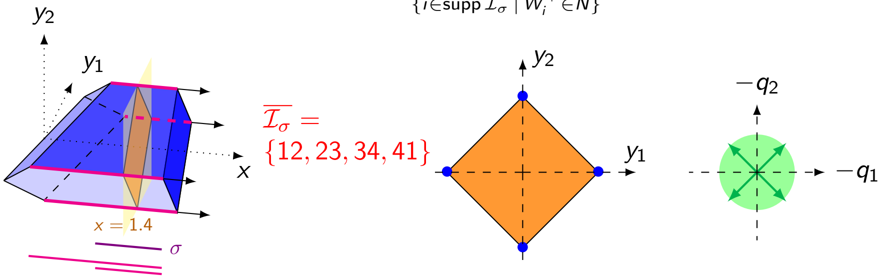
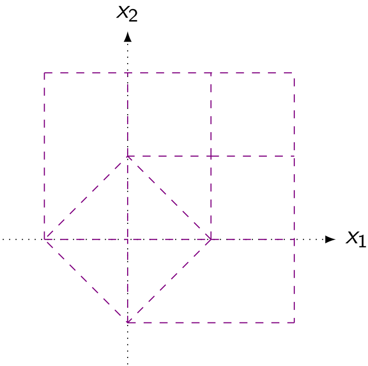




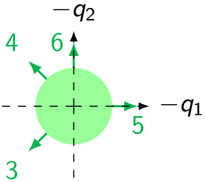
Simplex for 2SLP

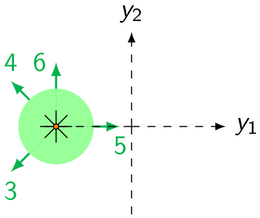
$$y_1 + y_2 \leqslant 1 \quad (1)$$
$$y_1 - y_2 \leqslant 1 \quad (2)$$
$$-y_1 - y_2 \leqslant 1 \quad (3)$$
$$-y_1 + y_2 \leqslant 1 \quad (4)$$
$$y_1 \leqslant x_1 \quad (5)$$
$$y_2 \leqslant x_2 \quad (6)$$
$$x_1 \leqslant 2 \quad (7)$$
$$x_2 \leqslant 2 \quad (8)$$

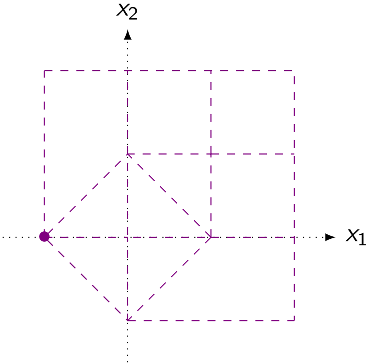


Simplex for 2SLP

$\overline{\mathcal{I}} = \{3456\}$

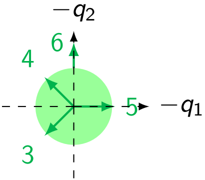


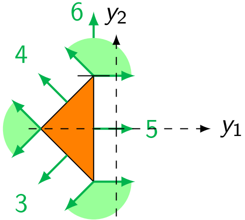


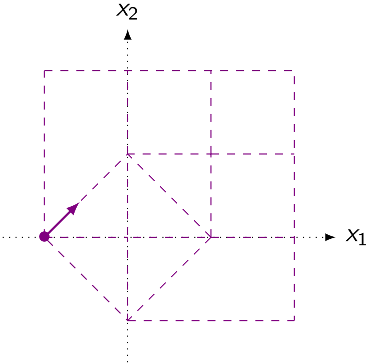


Simplex for 2SLP

$\overline{\mathcal{I}} = \{34, 35, 456\}$

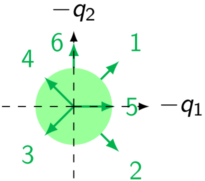


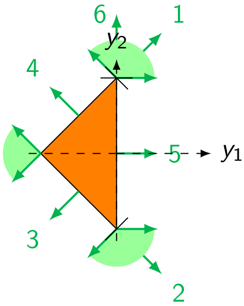


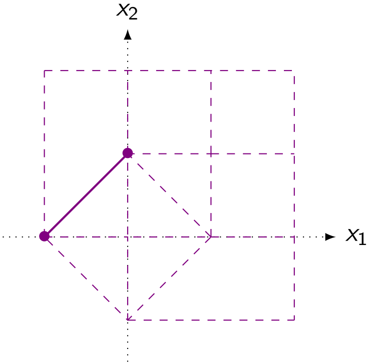


Simplex for 2SLP

$$\overline{\mathcal{I}} = \{34, 235, 1456\}$$

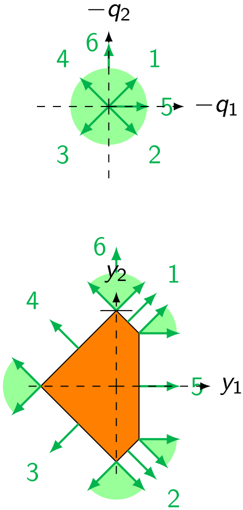


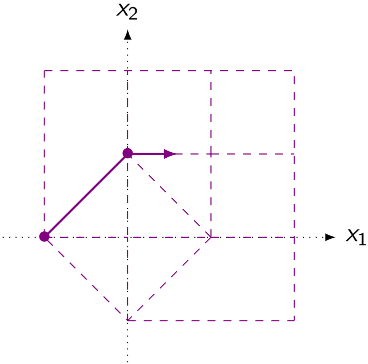




Simplex for 2SLP

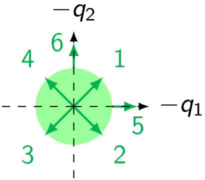
$\overline{\mathcal{I}} = \{34, 23, 25, 146, 15\}$

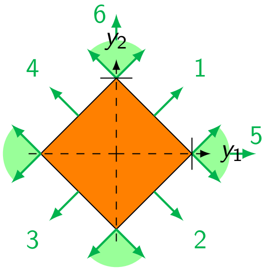


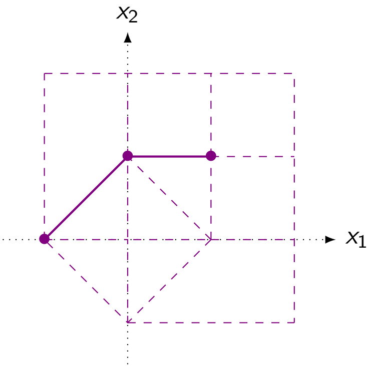


Simplex for 2SLP

$\overline{\mathcal{I}} = \{34, 23, 125, 146\}$

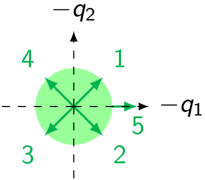


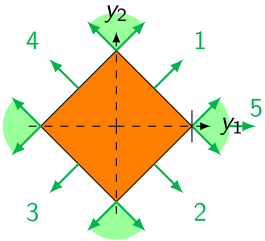


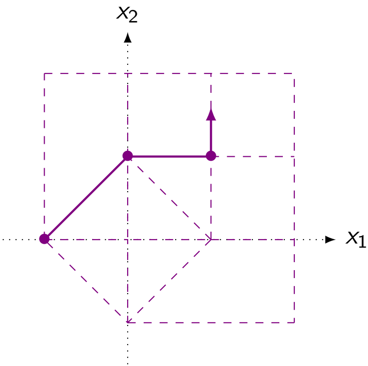


Simplex for 2SLP

$$\overline{\mathcal{I}} = \{34, 23, 125, 14\}$$

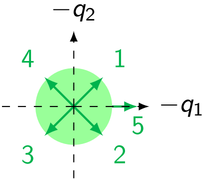


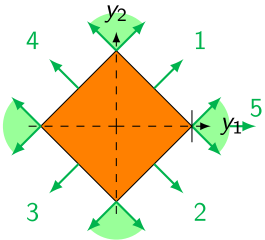


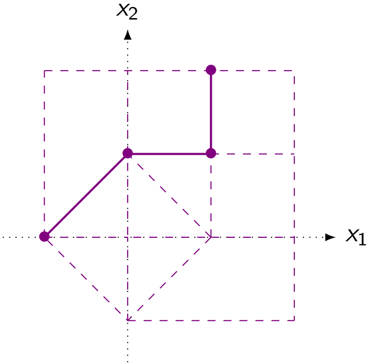


Simplex for 2SLP

$\overline{\mathcal{I}} = \{348, 238, 1258, 148\}$

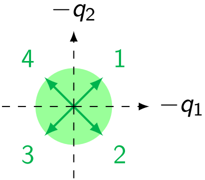


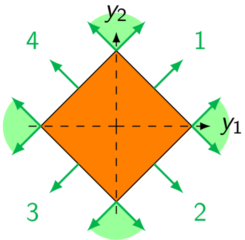


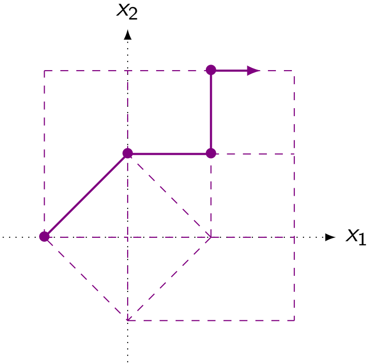


Simplex for 2SLP

$\overline{\mathcal{I}} = \{348, 238, 128, 148\}$







Simplex for 2SLP

$\overline{\mathcal{I}} = \{3478, 2378, 1278, 1478\}$