# The polyhedral structure and complexity of multistage stochastic linear problem with general cost distribution

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# Multistage stochastic linear programming (MSLP)

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\begin{aligned}
\min_{(\mathbf{x}_t)_{t\in[t_{\max}]}} \ & \mathbb{E}\Big[\sum_{t=1}^{t_{\max}} \mathbf{c}_t^\top \mathbf{x}_t\Big] & \\
\text{s.t.} \ & \mathbf{T}_t \mathbf{x}_{t-1} + \mathbf{W}_t \mathbf{x}_t \leqslant \mathbf{h}_t & \forall t \in [t_{\max}] \\
& \mathbf{x}_t \text{ random variable in } \mathbb{R}^{n_t} & \forall t \in [t_{\max}] \\
& \mathbf{x}_t \in \sigma(\mathbf{c}_k, \mathbf{T}_k, \mathbf{W}_k, \mathbf{h}_k)_{k\leqslant t} & \forall t \in [t_{\max}] \\
& \mathbf{x}_0 \equiv x_0 \text{ given} &
\end{aligned}
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where $\mathbf{c}_t \in \mathbb{R}^{n_t}$, $\mathbf{T}_t \in \mathbb{R}^{q_t \times n_{t-1}}$,$\mathbf{W}_t \in \mathbb{R}^{q_t \times n_t}$ and $\mathbf{h}_t \in \mathbb{R}^{q_t}$ are given random variables.

$(\mathbf{c}_t, \mathbf{T}_t, \mathbf{W}_t, \mathbf{h}_t)_{t\in[t_{\max}]}$ is an independent sequence.

We set $V_{t_{\max}+1} \equiv 0$ and:

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V_t(x_{t-1}) := \mathbb{E}\left[\begin{aligned} \min_{x_t \in \mathbb{R}^{n_t}} \ & \mathbf{c}_t^\top x_t + V_{t+1}(x_t) \\ \text{s.t.} \ & \mathbf{T}_t x_{t-1} + \mathbf{W}_t x_t \leqslant \mathbf{h}_t \end{aligned}\right]
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# Is $V$ polyhedral ?

$$V(x) = \mathbb{E}\left[\begin{aligned} \min_{y \in \mathbb{R}^m} \quad & \mathbf{c}^\top y + R(y) \\ \text{s.t.} \quad & \mathbf{T}x + \mathbf{W}y \leqslant \mathbf{h} \end{aligned}\right] = \mathbb{E}\Big[\min_{y \in \mathbb{R}^m} \big(\mathbf{c}^\top y + R(y) + \mathbb{I}_{\mathbf{T}x + \mathbf{W}y \leqslant \mathbf{h}}\big)\Big]$$

Question : On which conditions on the random variable $\mathbf{c}$, $\mathbf{T}$, $\mathbf{W}$ and $\mathbf{h}$, is $V$ polyhedral ?

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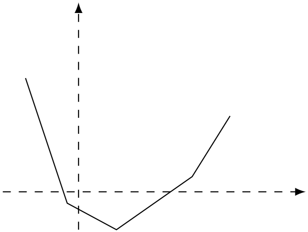

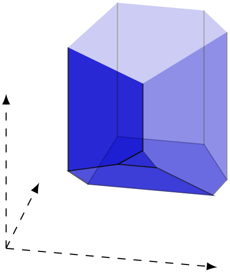

# We can assume $R \equiv 0$

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$$= \mathbb{E}\left[\begin{array}{ll} \min_{y \in \mathbb{R}^m, z \in \mathbb{R}} & \mathbf{c}^\top y + z \\ \text{s.t.} & \mathbf{T}x + \mathbf{W}y \leqslant \mathbf{h} \\ & (y, z) \in \text{epi}(R) \end{array}\right]$$

If $R$ is polyhedral, $\text{epi}(R) := \{(y, z) \mid Ay + b \leqslant z, Cy \leqslant d\}$

$\rightsquigarrow$ We may assume $R \equiv 0$ by setting

$$\widetilde{y} = \begin{pmatrix} y \\ z \end{pmatrix}, \widetilde{\mathbf{c}} = \begin{pmatrix} \mathbf{c} \\ 1 \end{pmatrix}, \widetilde{\mathbf{T}} = \begin{pmatrix} \mathbf{T} \\ 0 \end{pmatrix}, \widetilde{\mathbf{W}} = \begin{pmatrix} \mathbf{W} & 0 \\ A & -1 \\ C & 0 \end{pmatrix}, \widetilde{\mathbf{h}} = \begin{pmatrix} \mathbf{h} \\ -b \\ d \end{pmatrix}$$

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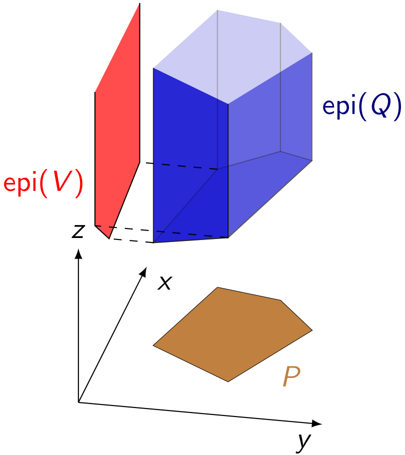
# $c$, $T$, $W$, $h$ deterministic $\Rightarrow$ $V$ polyhedral

For $x \in \mathbb{R}^n$,

$$V(x) = \min_{y \in \mathbb{R}^m} \left(c^\top y + \mathbb{I}_{Tx+Wy \leqslant h}\right)$$
$$= \min_{y \in \mathbb{R}^m} \left(c^\top y + \mathbb{I}_{(x,y) \in P}\right)$$
$$= \min_{y \in \mathbb{R}^m} Q(x,y)$$




$V$ is polyhedral because
epi($V$) $\subset \mathbb{R}^{n+1}$ is the projection of
epi($Q$) $\subset \mathbb{R}^{n+m+1}$ on $\mathbb{R}^{n+1}$.

# $\mathbf{c}$, $\mathbf{T}$, $\mathbf{W}$, $\mathbf{h}$ with finite support $\Rightarrow V$ polyhedral

**Theorem (see e.g. Shapiro, Dentcheva, Ruszczyński)**

*If* $\mathbf{c}$, $\mathbf{T}$, $\mathbf{W}$, $\mathbf{h}$ *have a finite support, then* $V$ *is polyhedral*

Proof:

$$
\begin{aligned}
V(x) &= \sum_{k=1}^{N} p_k V_k(x) \\
&= \sum_{k=1}^{N} p_k \min_{y \in \mathbb{R}^m} \left( c_k^\top y + \mathbb{I}_{T_k x + W_k y \leqslant h_k} \right)
\end{aligned}
$$

where $p_k := \mathbb{P}\big[(\mathbf{c}, \mathbf{T}, \mathbf{W}, \mathbf{h}) = (c_k, T_k, W_k, h_k)\big]$.
Each $V_k$ is polyhedral and $p_k \geqslant 0$.

$\rightsquigarrow$ Question: are these assumptions tight ?

# **c**, **T**, **W**, **h** with finite support ⇒ $V$ polyhedral

**Theorem (see e.g. Shapiro, Dentcheva, Ruszczyński)**

*If* **c**, **T**, **W**, **h** *have a finite support, then $V$ is polyhedral*

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$$V(x) = \sum_{k=1}^{N} p_k V_k(x)$$
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# Counter examples with stochastic constraints

Stochastic left hand side constraint $\mathbf{T}$

$$V(x) = \mathbb{E}\left[\begin{array}{ll} \min\limits_{y \in \mathbb{R}^m} & y \\ \text{s.t.} & \mathbf{u}x \leqslant y \\ & 1 \leqslant y \end{array}\right] = \mathbb{E}\big[\max(\mathbf{u}x, 1)\big] = \begin{cases} 1 & \text{if } x \leqslant 1 \\ \frac{x}{2} + \frac{1}{2x} & \text{if } x \geqslant 1 \end{cases}$$

Stochastic right hand side constraint $\mathbf{h}$

$$V(x) = \mathbb{E}\left[\begin{array}{ll} \min\limits_{y \in \mathbb{R}^m} & y \\ \text{s.t.} & \mathbf{u} \leqslant y \\ & x \leqslant y \end{array}\right] = \mathbb{E}\big[\max(x, \mathbf{u})\big] = \begin{cases} \frac{1}{2} & \text{if } x \leqslant 0 \\ \frac{x^2+1}{2} & \text{if } x \in [0,1] \\ x & \text{if } x \geqslant 1 \end{cases}$$

where $\mathbf{u}$ is uniform on $[0, 1]$.

# Remaining case: only $\mathbf{c}$ stochastic

$$V(x) = \mathbb{E}\left[\begin{array}{ll}\min_{y\in\mathbb{R}^m} & \mathbf{c}^\top y \\ \text{s.t.} & Tx + Wy \leqslant h\end{array}\right] = \mathbb{E}\left[\min_{y\in\mathbb{R}^m}\left(\mathbf{c}^\top y + \mathbb{I}_{Tx+Wy\leqslant h}\right)\right]$$

**Theorem (FGL 2020)**

*If $T$, $W$ and $h$ are deterministic, then for all distributions of $\mathbf{c}$ such that $V$ is well defined, $V$ is polyhedral.*

$\rightsquigarrow$ This extends easily to $\mathbf{T}$, $\mathbf{W}$ and $\mathbf{h}$ with a finite support.

Let's dive in !

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For a given $x$,

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where

$$P_x := \{y \in \mathbb{R}^m \mid Tx + Wy \leqslant h\}$$

Illustrative running example:

$$P_x := \{y \in \mathbb{R}^m \mid \|y\|_1 \leqslant 1, \quad y_1 \leqslant x, \quad y_2 \leqslant x\}$$




$P_x$ for $x = 0.8$




$P_x$ for $x = 0.3$

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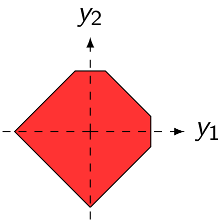


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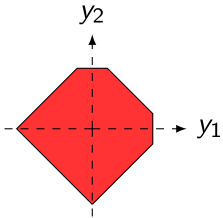


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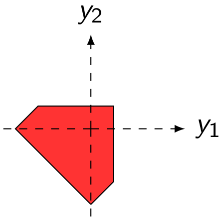


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# Normal fan $\mathcal{N}(P_x)$

**Definition**

*The normal fan of the fiber* $P_x$ *is*

$$\mathcal{N}(P_x) := \{N_{P_x}(y) \mid y \in P_x\}$$

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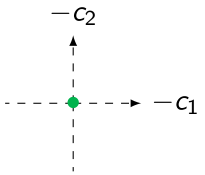


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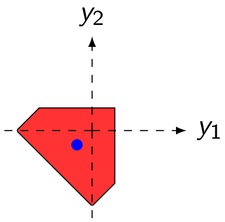


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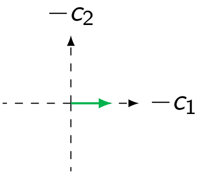


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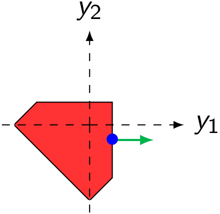


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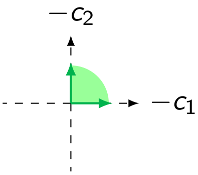


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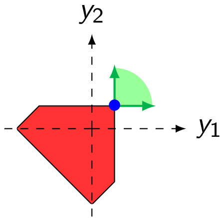


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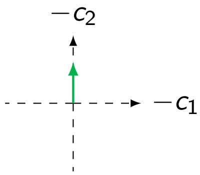


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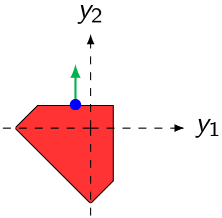


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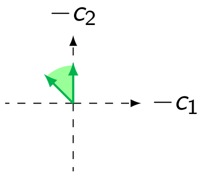


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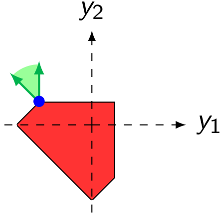


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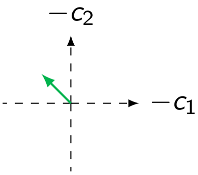
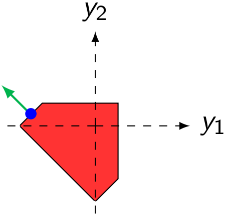



$N_{P_x}(y)$ for $x = 0.3$




$P_x$ $y$ and $N_{P_x}(y)$ for $x = 0.3$

# Normal fan $\mathcal{N}(P_x)$

**Definition**

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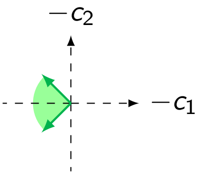
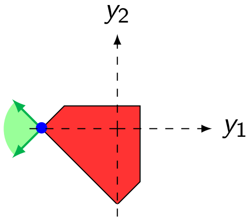



$N_{P_x}(y)$ for $x = 0.3$




$P_x$ $y$ and $N_{P_x}(y)$ for $x = 0.3$

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## Definition

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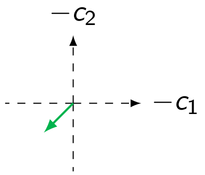
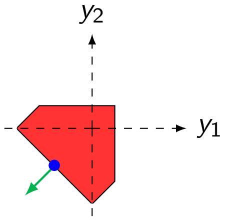



$N_{P_x}(y)$ for $x = 0.3$

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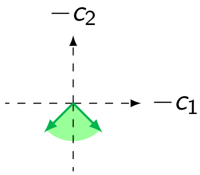
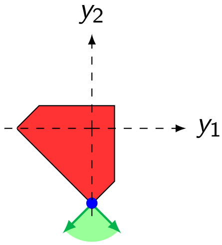



$N_{P_x}(y)$ for $x = 0.3$

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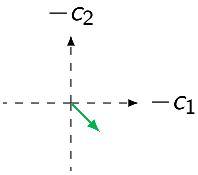
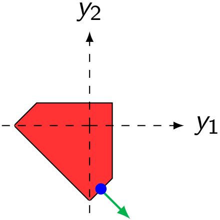



$N_{P_x}(y)$ for $x = 0.3$




$P_x$ $y$ and $N_{P_x}(y)$ for $x = 0.3$

# Normal fan $\mathcal{N}(P_x)$

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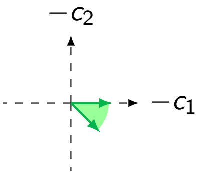
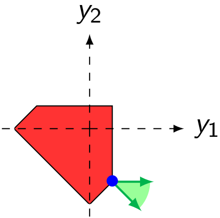



$N_{P_x}(y)$ for $x = 0.3$

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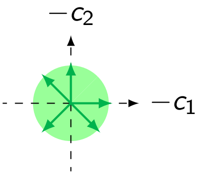
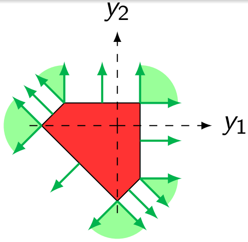
**Proposition**

*If $P_x$ is bounded, $\{\mathrm{ri}(N) \mid N \in \mathcal{N}(P_x)\}$ is a partition of $\mathbb{R}^m$.*




$\mathcal{N}(P_x)$ for $x = 0.3$




$P_x$ and $\mathcal{N}(P_x)$ for $x = 0.3$

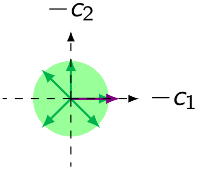
# $\mathcal{N}(P_x)$: partition of $-\mathbf{c}$ coherent with the min

For a given $x$, we have

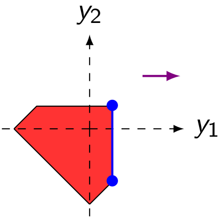
$$V(x) = \mathbb{E}\big[\min_{y \in P_x} \mathbf{c}^\top y\big]$$

For any $N \in \mathcal{N}(P_x)$ and $-c \to \arg\min_{y\in P_x} c^\top y$ is constant for all $-c \in \mathrm{ri}(N)$.

$\arg\min_{y\in P_x} c^\top y$ is a face of $P_x$.




Cost $-c$ and $\mathcal{N}(P_x)$ for $x = 0.3$




$P_x$ for $x = 0.3$

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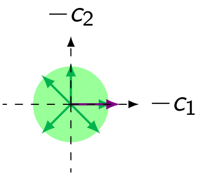
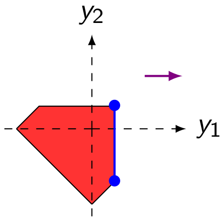



Cost $-c$ and $\mathcal{N}(P_x)$ for $x = 0.3$




$P_x$ for $x = 0.3$

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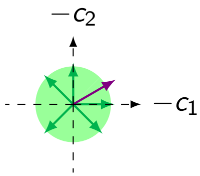
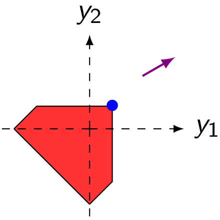



Cost $-c$ and $\mathcal{N}(P_x)$ for $x = 0.3$

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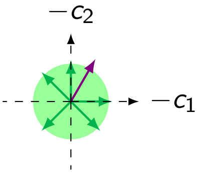
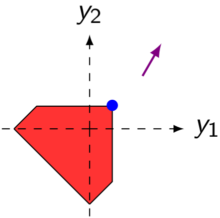



Cost $-c$ and $\mathcal{N}(P_x)$ for $x = 0.3$

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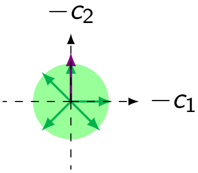
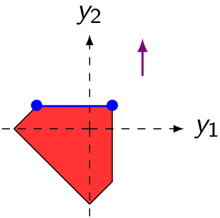



Cost $-c$ and $\mathcal{N}(P_x)$ for $x = 0.3$

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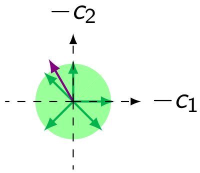
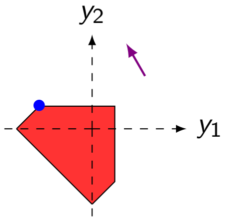



Cost $-c$ and $\mathcal{N}(P_x)$ for $x = 0.3$

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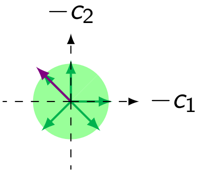
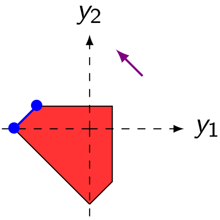



Cost $-c$ and $\mathcal{N}(P_x)$ for $x = 0.3$




$P_x$ for $x = 0.3$

## $\mathcal{N}(P_x)$: partition of $-\mathbf{c}$ coherent with the min

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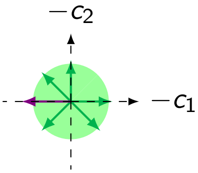
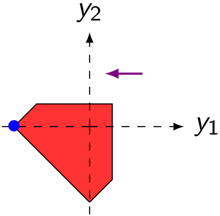



Cost $-c$ and $\mathcal{N}(P_x)$ for $x = 0.3$




$P_x$ for $x = 0.3$

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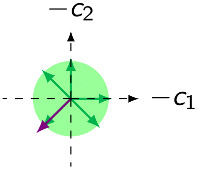
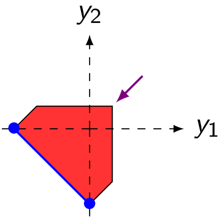



Cost $-c$ and $\mathcal{N}(P_x)$ for $x = 0.3$




$P_x$ for $x = 0.3$

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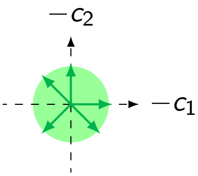
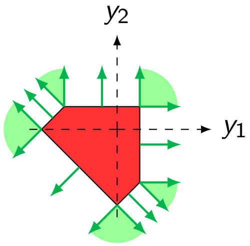



$\mathcal{N}(P_x)$ for $x = 0.3$



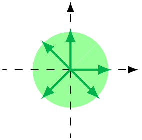

$P_x$ for $x = 0.3$

# Reduction to a finite sum

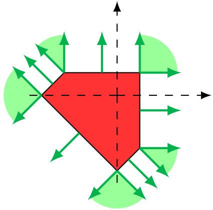
For a fixed $x$,

$$V(x) = \mathbb{E}\big[\min_{y \in P_x} \mathbf{c}^\top y\big] = \sum_{N \in \mathcal{N}(P_x)} \mathbb{E}\big[\mathbf{c}^\top \mathbb{1}_{\mathbf{c} \in -\operatorname{ri} N}\big] y_N(x)$$

where $y_N(x) \in \arg\min_{y \in P_x} c^\top y$ for any $c \in \operatorname{ri}(N)$.




$\mathcal{N}(P_x)$ for $x = 0.3$

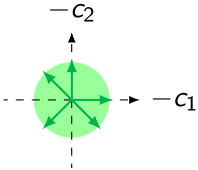



$P_x$ and $\mathcal{N}(P_x)$ for $x = 0.3$

# General cost $\mathbf{c}$ is equivalent to discrete cost $\check{\mathbf{c}}$ for given $x$

For a fixed $x$,

$$V(x) = \mathbb{E}\big[\min_{y \in P_x} \mathbf{c}^\top y\big]$$
$$= \sum_{N \in \mathcal{N}(P_x)} \mathbb{E}\big[\mathbf{c}^\top \mathbb{1}_{\mathbf{c} \in -\operatorname{ri} N}\big] y_N(x)$$



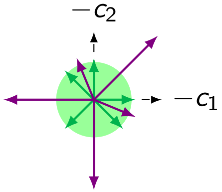

$\mathcal{N}(P_x)$ for $x = 0.3$

We draw a continuous cost $\mathbf{c}$.

# General cost $\mathbf{c}$ is equivalent to discrete cost $\check{\mathbf{c}}$ for given $x$

For a fixed $x$,

$$
\begin{aligned}
V(x) &= \mathbb{E}\big[\min_{y \in P_x} \mathbf{c}^\top y\big] \\
&= \sum_{N \in \mathcal{N}(P_x)} \mathbb{E}\big[\mathbf{c}^\top \mathbb{1}_{\mathbf{c} \in -\operatorname{ri} N}\big] y_N(x) \\
&= \sum_{N \in \mathcal{N}(P_x)} p_N \check{c}_N y_N(x)
\end{aligned}
$$




$\mathcal{N}(P_x)$ and $p_N \check{c}_N$ for $x = 0.3$

For $N \in \mathcal{N}(P_x)$,

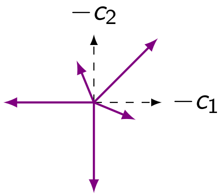
$$
\begin{aligned}
p_N &:= \mathbb{P}\big[\mathbf{c} \in -\operatorname{ri} N\big] \\
\check{c}_N &:= \mathbb{E}\big[\mathbf{c} | \mathbf{c} \in -\operatorname{ri} N\big]
\end{aligned}
$$

Instead of drawing a general $\mathbf{c}$, we draw a discrete cost $\check{\mathbf{c}}$ indexed by the finite collection $\mathcal{N}(P_x)$.

# General cost $\mathbf{c}$ is equivalent to discrete cost $\check{\mathbf{c}}$ for given $x$

For a fixed $x$,

$$V(x) = \mathbb{E}\big[\min_{y \in P_x} \mathbf{c}^\top y\big]$$
$$= \sum_{N \in \mathcal{N}(P_x)} \mathbb{E}\big[\mathbf{c}^\top \mathbb{1}_{\mathbf{c} \in -\operatorname{ri} N}\big] y_N(x)$$
$$= \sum_{N \in \mathcal{N}(P_x)} p_N \check{c}_N y_N(x)$$
$$= \sum_{N \in \mathcal{N}(P_x)} p_N \min_{y \in P_x} \check{c}_N{}^\top y$$




$p_N \check{c}_N$ for $x = 0.3$

For $N \in \mathcal{N}(P_x)$,

$$p_N := \mathbb{P}\big[\mathbf{c} \in -\operatorname{ri} N\big]$$
$$\check{c}_N := \mathbb{E}\big[\mathbf{c} | \mathbf{c} \in -\operatorname{ri} N\big]$$

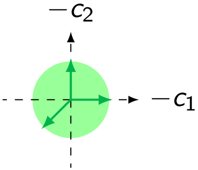
Instead of drawing a general $\mathbf{c}$, we draw a discrete cost $\check{\mathbf{c}}$ indexed by the finite collection $\mathcal{N}(P_x)$.

# Contents

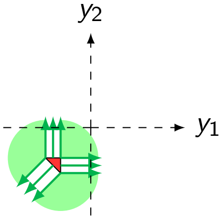
# $\mathcal{N}(P_x)$ is piecewise constant with $x$.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

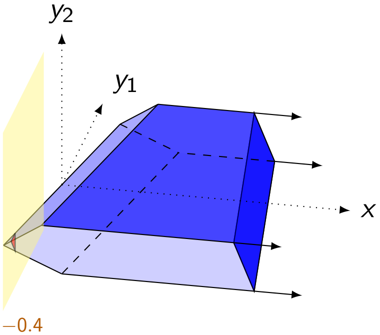
$$x = -0.4$$




$\mathcal{N}(P_x)$




$P_x$ and $\mathcal{N}(P_x)$

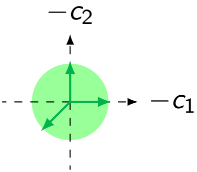



$P$ and $P_x$

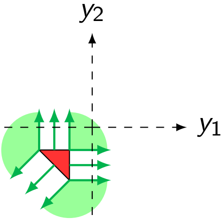
# $\mathcal{N}(P_x)$ is piecewise constant with $x$.

$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

$$x = -0.3$$




$\mathcal{N}(P_x)$

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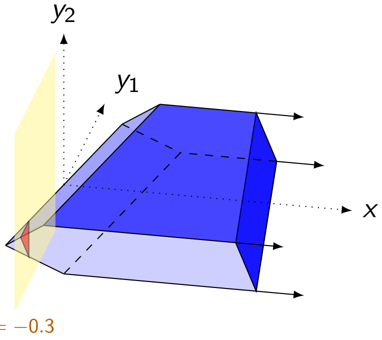


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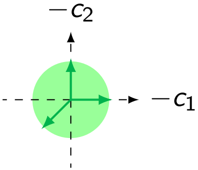
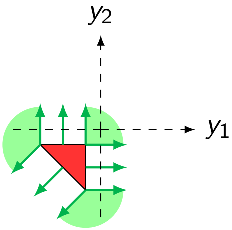
# $\mathcal{N}(P_x)$ is piecewise constant with $x$.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

$$x = -0.2$$




$\mathcal{N}(P_x)$

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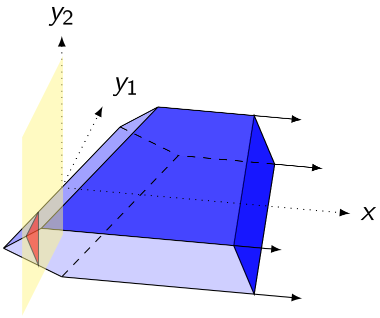


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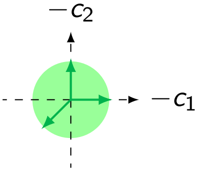
$$x = -0.1$$

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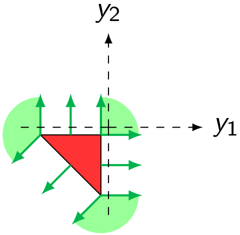


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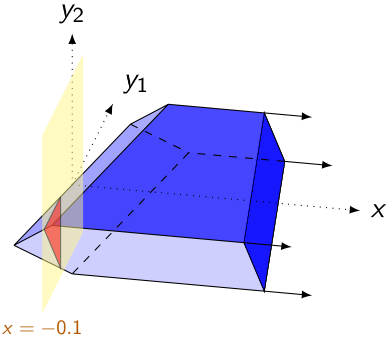


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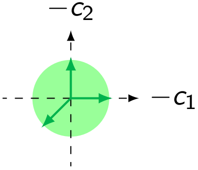
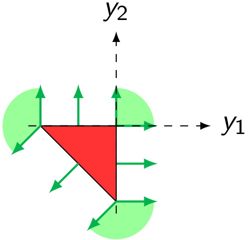
$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

$$x = 0$$




$\mathcal{N}(P_x)$

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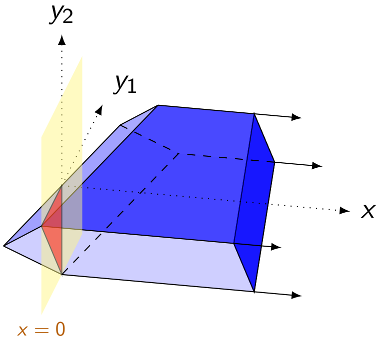


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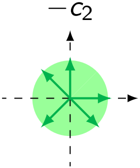


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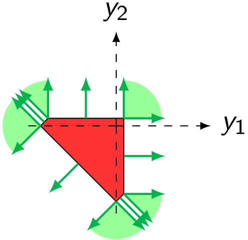


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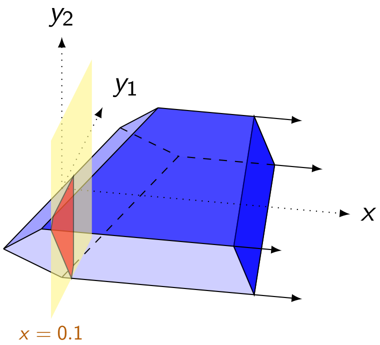


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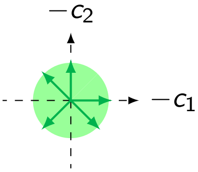
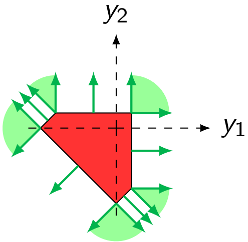



$\mathcal{N}(P_x)$

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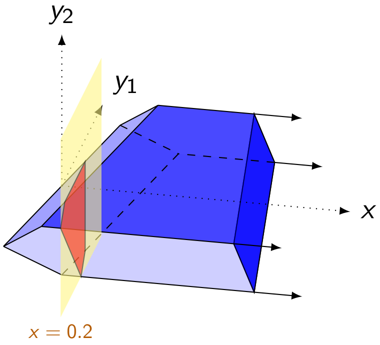


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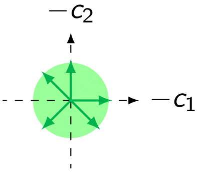
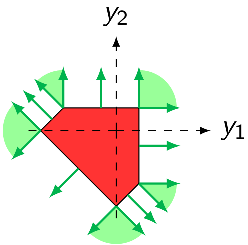
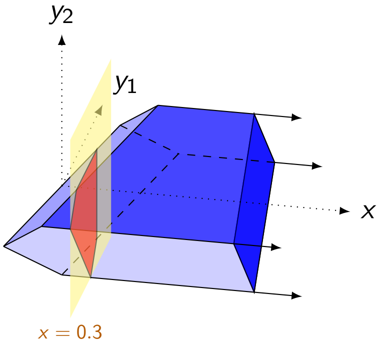
$x = 0.3$




$\mathcal{N}(P_x)$




$P_x$ and $\mathcal{N}(P_x)$

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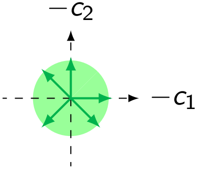
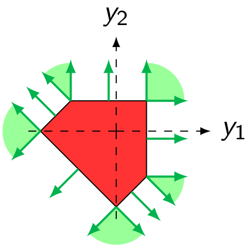
$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

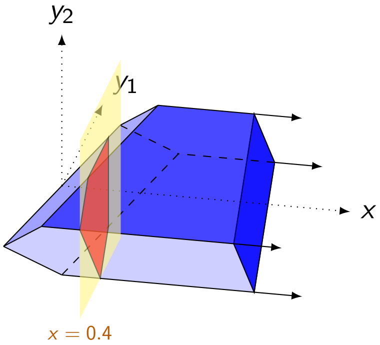
$$x = 0.4$$




$\mathcal{N}(P_x)$




$P_x$ and $\mathcal{N}(P_x)$

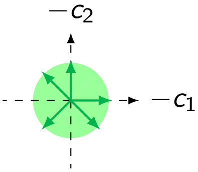



$P$ and $P_x$

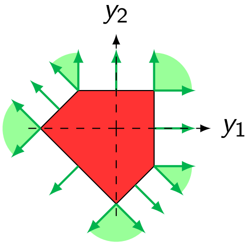
# $\mathcal{N}(P_x)$ is piecewise constant with $x$.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

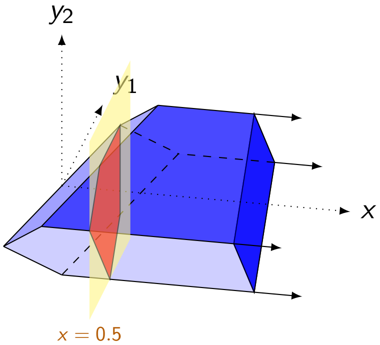
$$x = 0.5$$




$\mathcal{N}(P_x)$




$P_x$ and $\mathcal{N}(P_x)$

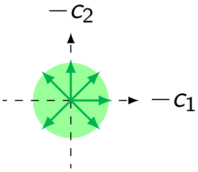



$P$ and $P_x$

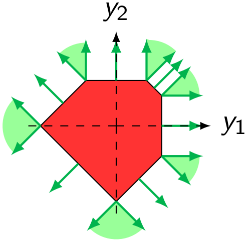
# $\mathcal{N}(P_x)$ is piecewise constant with $x$.

$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

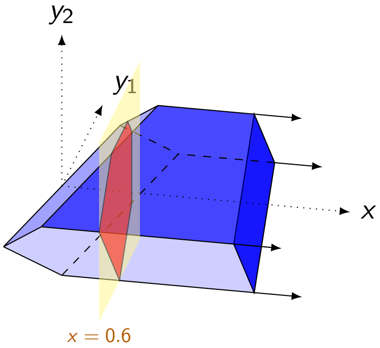
$$x = 0.6$$




$\mathcal{N}(P_x)$




$P_x$ and $\mathcal{N}(P_x)$

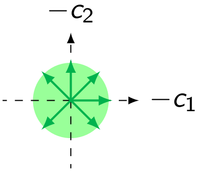



$P$ and $P_x$

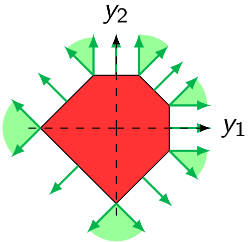
# $\mathcal{N}(P_x)$ is piecewise constant with $x$.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

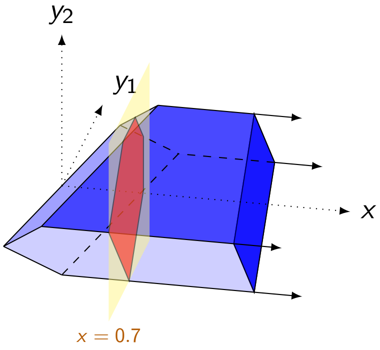
$x = 0.7$




$\mathcal{N}(P_x)$




$P_x$ and $\mathcal{N}(P_x)$

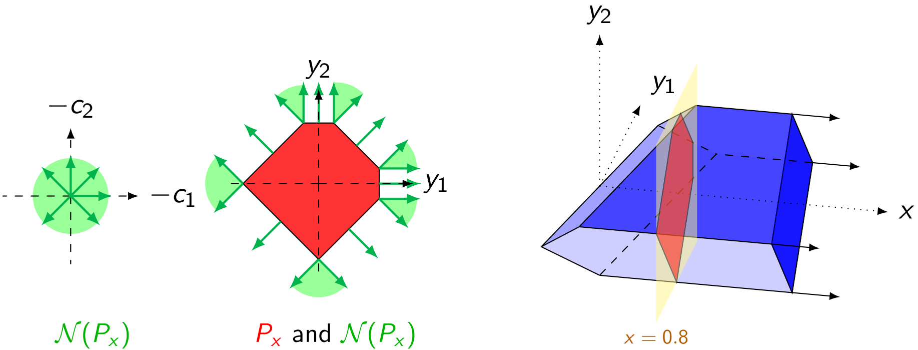



$P$ and $P_x$

# $\mathcal{N}(P_x)$ is piecewise constant with $x$.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

$x = 0.8$

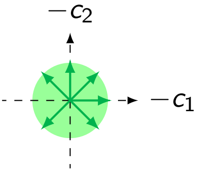



$\mathcal{N}(P_x)$ $P_x$ and $\mathcal{N}(P_x)$ $P$ and $P_x$

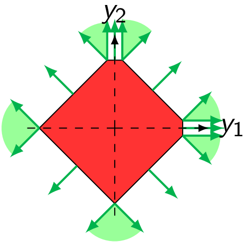
# $\mathcal{N}(P_x)$ is piecewise constant with $x$.

$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

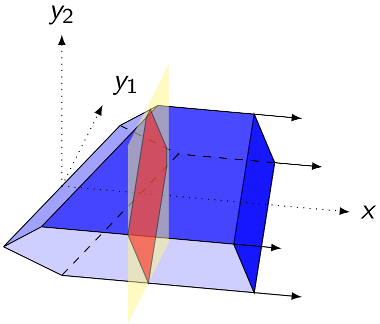
$x = 0.9$




$\mathcal{N}(P_x)$




$P_x$ and $\mathcal{N}(P_x)$

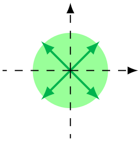



$P$ and $P_x$

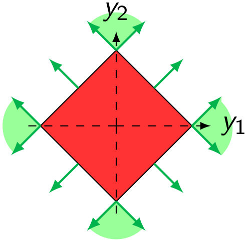
# $\mathcal{N}(P_x)$ is piecewise constant with $x$.

$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

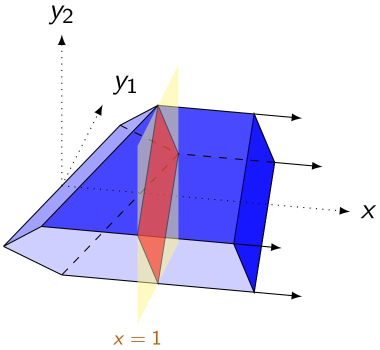
$x = 1$




$\mathcal{N}(P_x)$




$P_x$ and $\mathcal{N}(P_x)$

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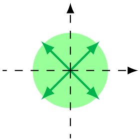
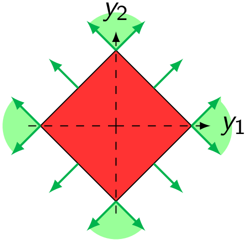
$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

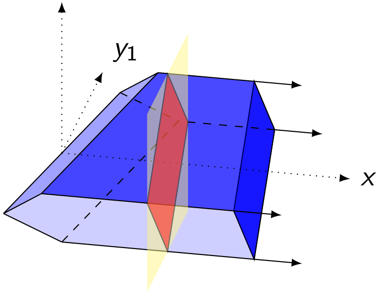
$x = 1.1$




$\mathcal{N}(P_x)$




$P_x$ and $\mathcal{N}(P_x)$

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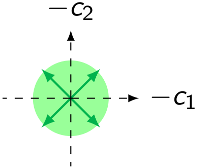
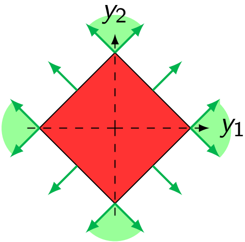
$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

$$x = 1.2$$




$\mathcal{N}(P_x)$

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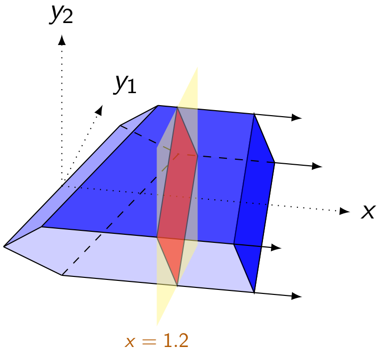


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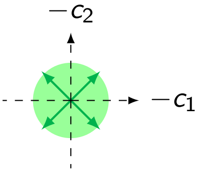
$x = 1.3$

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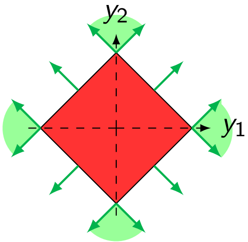


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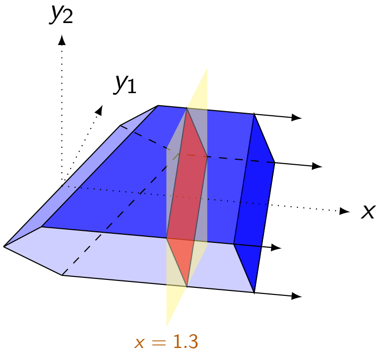


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$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

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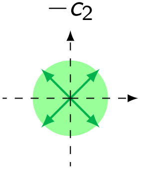


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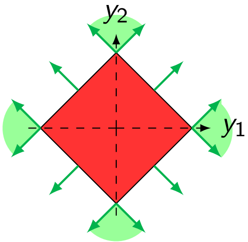


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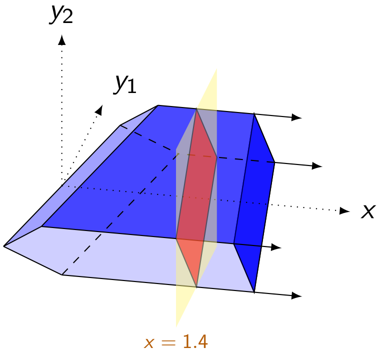


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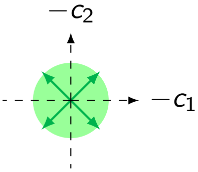
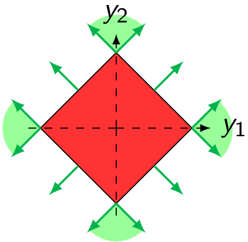
$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

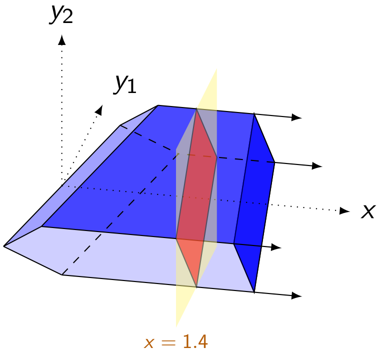
$$x = 1.4$$




$\mathcal{N}(P_x)$




$P_x$ and $\mathcal{N}(P_x)$

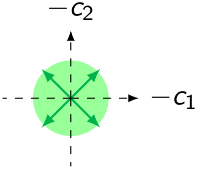



$P$ and $P_x$

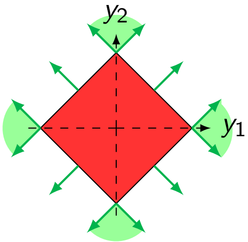
# $\mathcal{N}(P_x)$ is piecewise constant with $x$.

$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

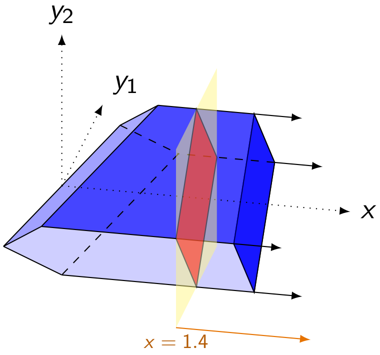
$$x = 1.4$$




$\mathcal{N}(P_x)$




$P_x$ and $\mathcal{N}(P_x)$

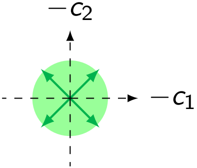



$P$ and $P_x$

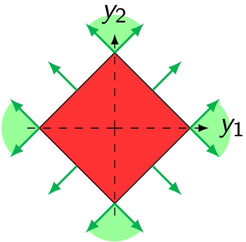
# $\mathcal{N}(P_x)$ is piecewise constant with $x$.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

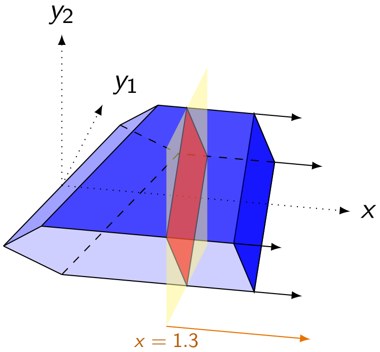
$x = 1.3$




$\mathcal{N}(P_x)$




$P_x$ and $\mathcal{N}(P_x)$

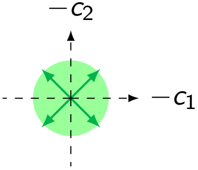



$P$ and $P_x$

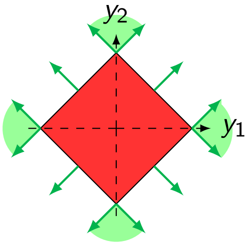
# $\mathcal{N}(P_x)$ is piecewise constant with $x$.

$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

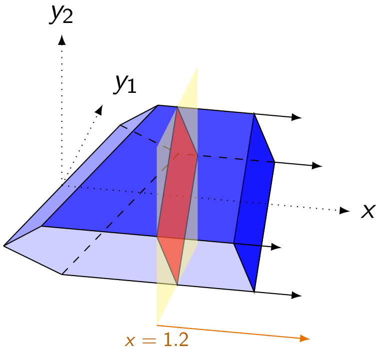
$$x = 1.2$$




$\mathcal{N}(P_x)$




$P_x$ and $\mathcal{N}(P_x)$

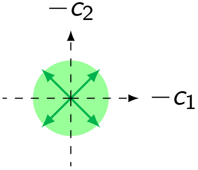



$P$ and $P_x$

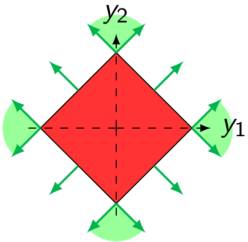
# $\mathcal{N}(P_x)$ is piecewise constant with $x$.

$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

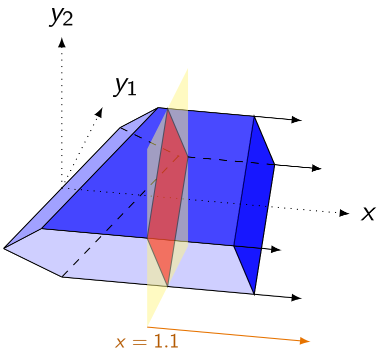
$x = 1.1$




$\mathcal{N}(P_x)$




$P_x$ and $\mathcal{N}(P_x)$

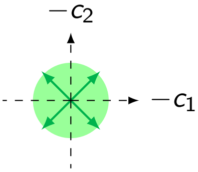



$P$ and $P_x$

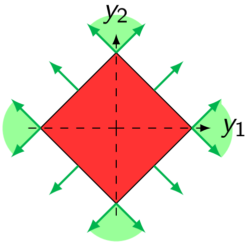
# $\mathcal{N}(P_x)$ is piecewise constant with $x$.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

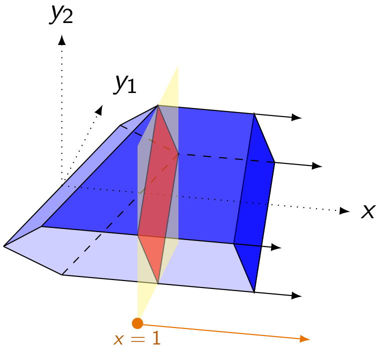
$x = 1$




$\mathcal{N}(P_x)$




$P_x$ and $\mathcal{N}(P_x)$

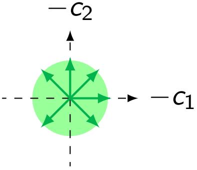



$P$ and $P_x$

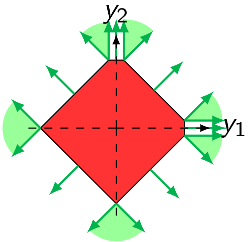
# $\mathcal{N}(P_x)$ is piecewise constant with $x$.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

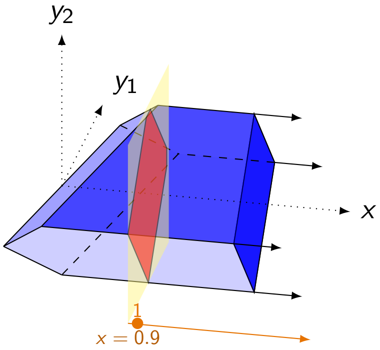
$x = 0.9$




$\mathcal{N}(P_x)$




$P_x$ and $\mathcal{N}(P_x)$

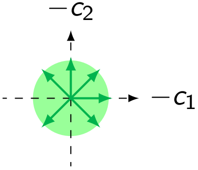



$P$ and $P_x$

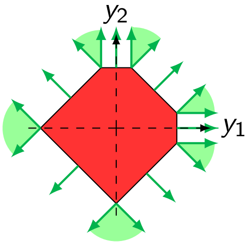
# $\mathcal{N}(P_x)$ is piecewise constant with $x$.

$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

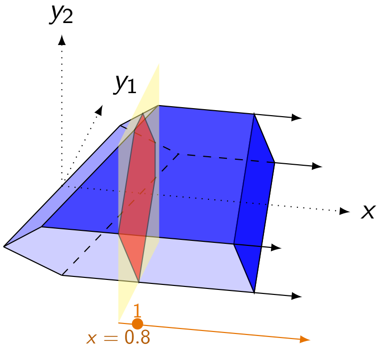
$x = 0.8$




$\mathcal{N}(P_x)$




$P_x$ and $\mathcal{N}(P_x)$

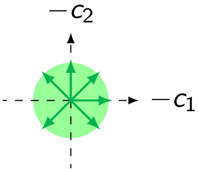



$P$ and $P_x$

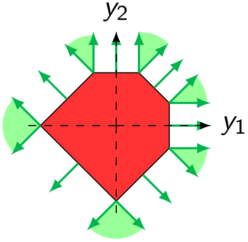
# $\mathcal{N}(P_x)$ is piecewise constant with $x$.

$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

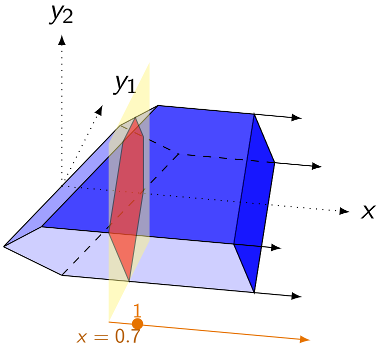
$x = 0.7$




$\mathcal{N}(P_x)$




$P_x$ and $\mathcal{N}(P_x)$

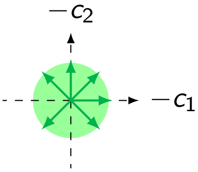



$P$ and $P_x$

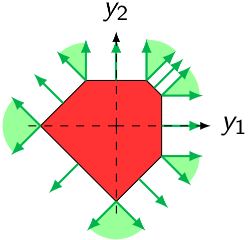
# $\mathcal{N}(P_x)$ is piecewise constant with $x$.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

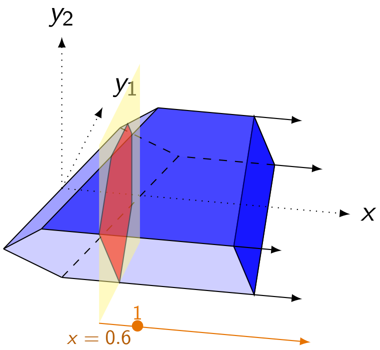
$$x = 0.6$$




$\mathcal{N}(P_x)$




$P_x$ and $\mathcal{N}(P_x)$

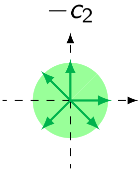



$P$ and $P_x$

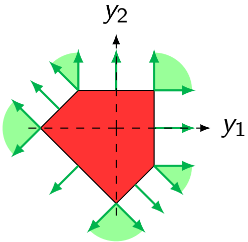
# $\mathcal{N}(P_x)$ is piecewise constant with $x$.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

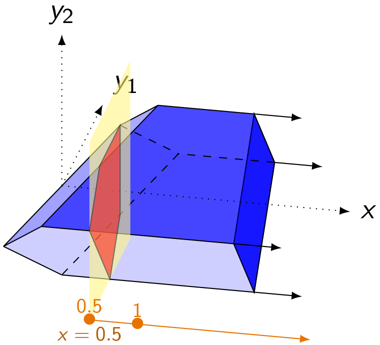
$x = 0.5$




$\mathcal{N}(P_x)$




$P_x$ and $\mathcal{N}(P_x)$

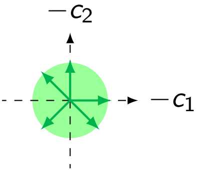



$P$ and $P_x$

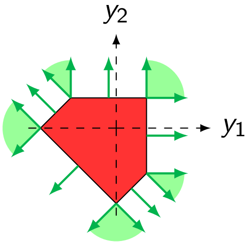
# $\mathcal{N}(P_x)$ is piecewise constant with $x$.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

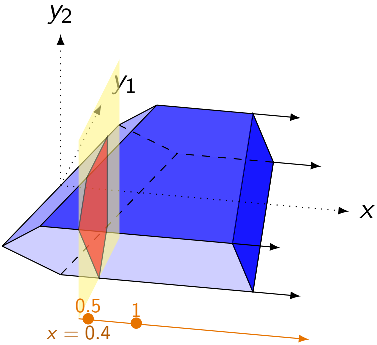
$$x = 0.4$$




$\mathcal{N}(P_x)$




$P_x$ and $\mathcal{N}(P_x)$

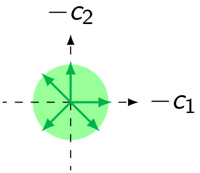



$P$ and $P_x$

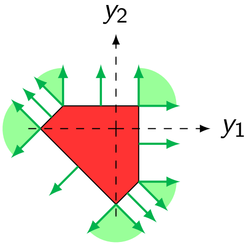
# $\mathcal{N}(P_x)$ is piecewise constant with $x$.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

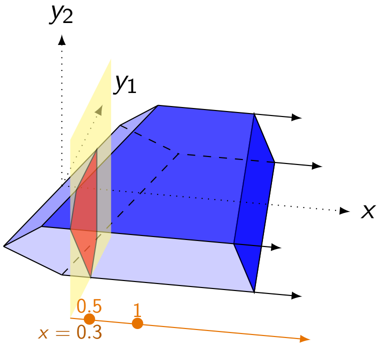
$$x = 0.3$$




$\mathcal{N}(P_x)$




$P_x$ and $\mathcal{N}(P_x)$

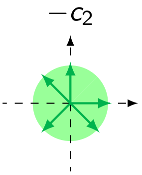



$P$ and $P_x$

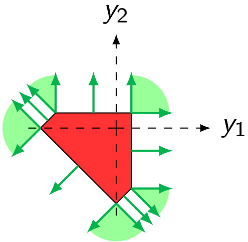
# $\mathcal{N}(P_x)$ is piecewise constant with $x$.

$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

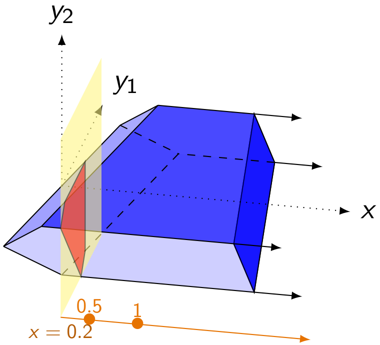
$$x = 0.2$$




$\mathcal{N}(P_x)$




$P_x$ and $\mathcal{N}(P_x)$

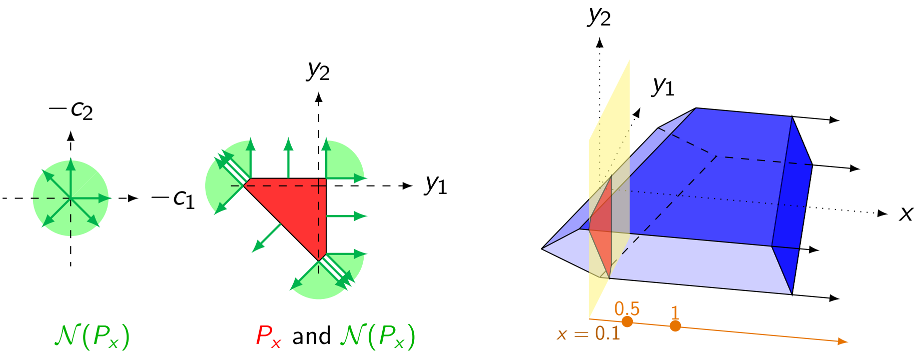



$P$ and $P_x$

# $\mathcal{N}(P_x)$ is piecewise constant with $x$.

$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

$$x = 0.1$$

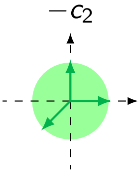



$\mathcal{N}(P_x)$ — $P_x$ and $\mathcal{N}(P_x)$ — $P$ and $P_x$

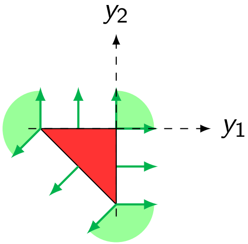
# $\mathcal{N}(P_x)$ is piecewise constant with $x$.

$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

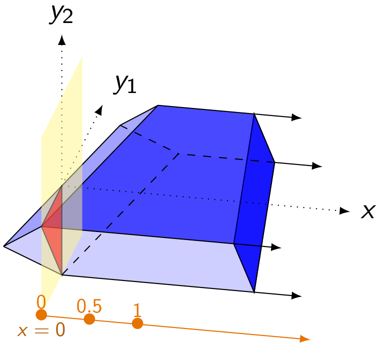
$x = 0$




$\mathcal{N}(P_x)$




$P_x$ and $\mathcal{N}(P_x)$

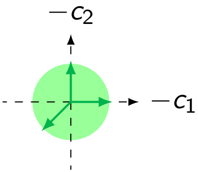



$P$ and $P_x$

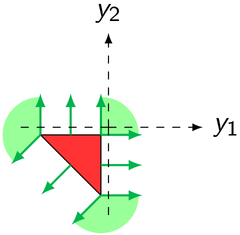
# $\mathcal{N}(P_x)$ is piecewise constant with $x$.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

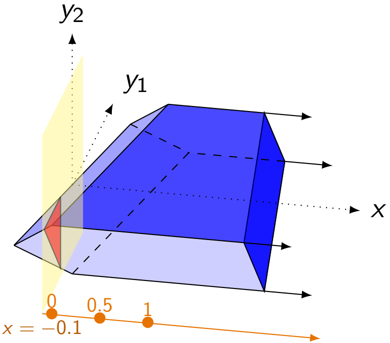
$$x = -0.1$$




$\mathcal{N}(P_x)$




$P_x$ and $\mathcal{N}(P_x)$

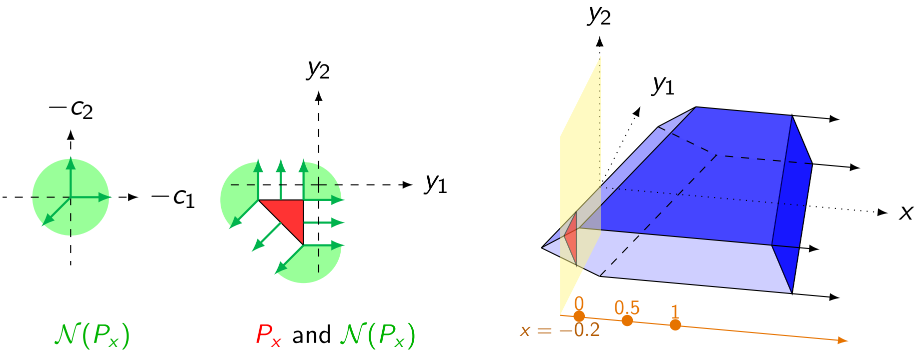



$P$ and $P_x$

# $\mathcal{N}(P_x)$ is piecewise constant with $x$.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

$$x = -0.2$$




$\mathcal{N}(P_x)$

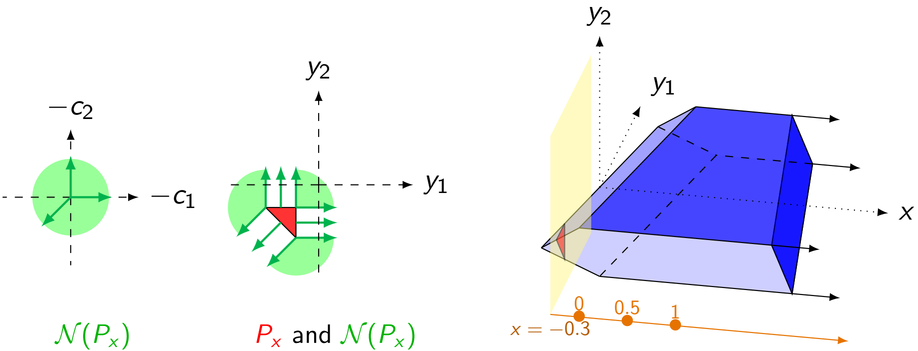
$P_x$ and $\mathcal{N}(P_x)$

$P$ and $P_x$

# $\mathcal{N}(P_x)$ is piecewise constant with $x$.

$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

$$x = -0.3$$

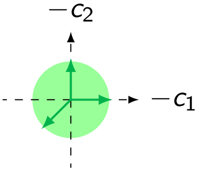



$\mathcal{N}(P_x)$ $\quad$ $P_x$ and $\mathcal{N}(P_x)$ $\quad$ $P$ and $P_x$

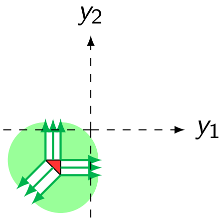
# $\mathcal{N}(P_x)$ is piecewise constant with $x$.

$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

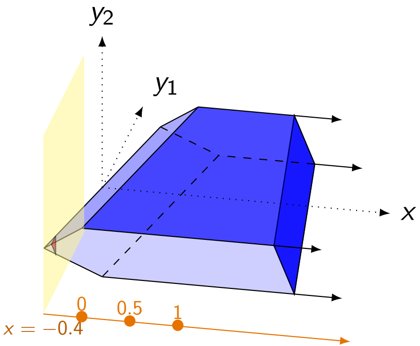
$x = -0.4$




$\mathcal{N}(P_x)$




$P_x$ and $\mathcal{N}(P_x)$



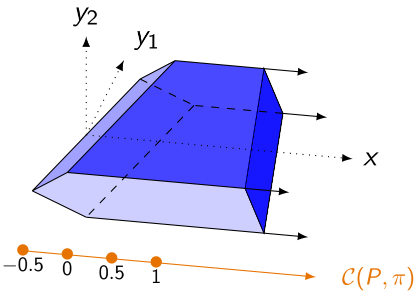

$P$ and $P_x$

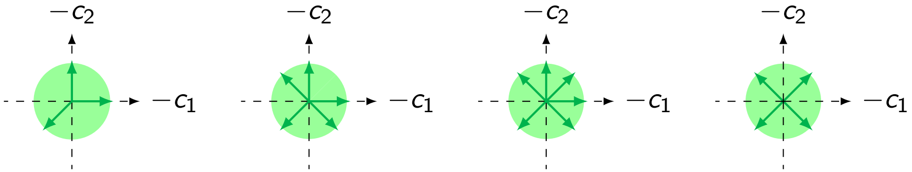
# What are the constant regions of $\mathcal{N}(P_x)$ ?

**Lemma**

*There exists a collection* $\mathcal{C}(P,\pi)$ *called the* **chamber complex** *whose relative interior of cells are the constant regions of* $x \to \mathcal{N}(P_x)$.

*For* $\sigma \in \mathcal{C}(P,\pi)$ *and* $x, x' \in \mathrm{ri}(\sigma)$, $\mathcal{N}(P_x) = \mathcal{N}(P_{x'}) =: \mathcal{N}_\sigma$




$\mathcal{N}_\sigma$ for $\sigma = [-0.5, 0]$ $\mathcal{N}_\sigma$ for $\sigma = [0, 0.5]$ $\mathcal{N}_\sigma$ for $\sigma = [0.5, 1]$ $\mathcal{N}_\sigma$ for $\sigma = [1, +\infty)$

# Chamber complex

**Definition (Billera, Sturmfels 92)**

The *chamber complex* $\mathcal{C}(P,\pi)$ of $P$ along $\pi$ is

$$\mathcal{C}(P,\pi) := \{\sigma_{P,\pi}(x) \mid x \in \pi(P)\}$$

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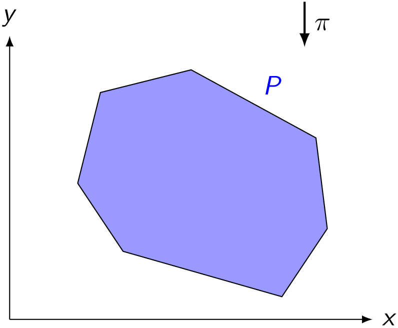
$$\sigma_{P,\pi}(x) := \bigcap_{F \in \mathcal{F}(P) \text{ s.t. } x \in \pi(F)} \pi(F)$$




where $\mathcal{F}(P)$ is the set of faces of $P$ and $\pi$ is the projection $(x,y) \to x$

$$\pi(E) := \{x \in \mathbb{R}^n \mid \exists y \in \mathbb{R}^m,\ (x,y) \in E\}$$

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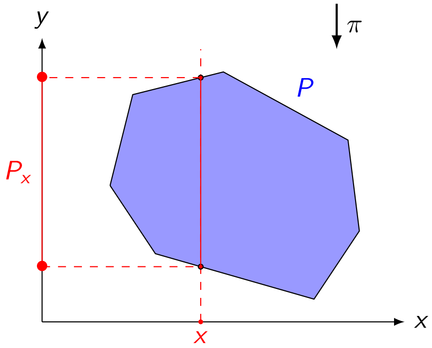


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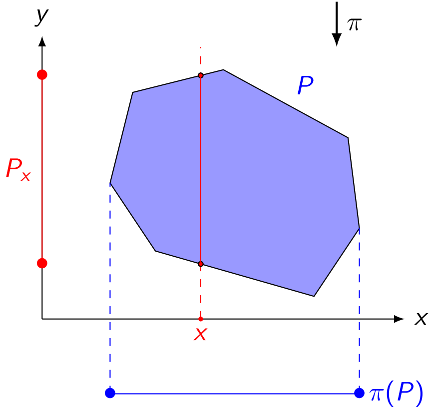


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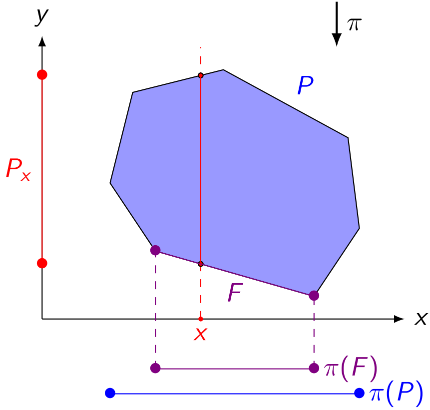


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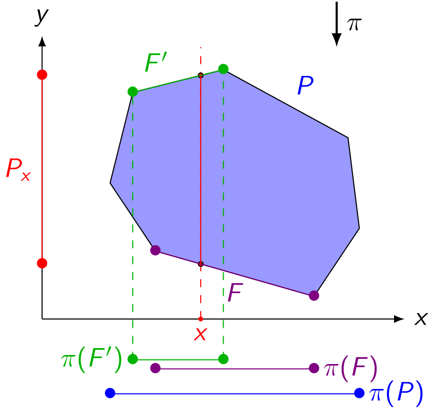


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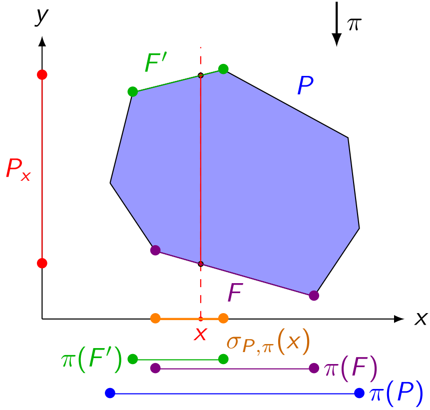


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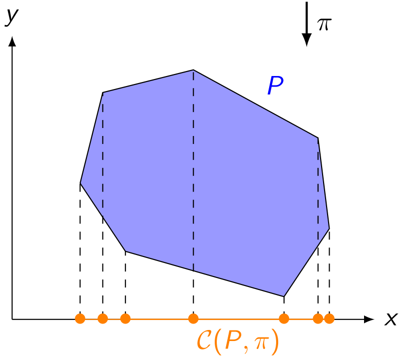


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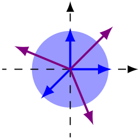

$\mathcal{N}_\sigma$ and $\check{\mathbf{c}}$

For all $x \in \mathrm{ri}(\sigma)$,

$$V(x) = \sum_{N \in \mathcal{N}_\sigma} p_N \min_{y \in P_x} \check{c}_N{}^\top y$$

For all $x \in \mathrm{ri}(\tau)$,

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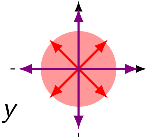

$\mathcal{N}_\tau$ and $\check{\mathbf{c}}$

**Theorem (Quantization of the cost distribution)**

*Let $\mathcal{R} = \bigwedge_{\sigma \in \mathcal{C}(P,\pi)} -\mathcal{N}_\sigma$, then* **for all** $x \in \mathbb{R}^n$

$$V(x) = \sum_{R \in \mathcal{R}} \check{p}_R \min_{y \in \mathbb{R}^m} \check{c}_R^\top y + \mathbb{I}_{Tx+Wy \leqslant h}$$

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Bonus: This quantization method works for *every distribution of* $\mathbf{c}$ !

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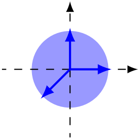
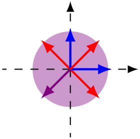
We take the *common refinement*:

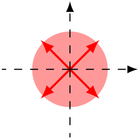
$$\mathcal{R} := \mathcal{N}_\sigma \wedge \mathcal{N}_\tau = \{N \cap N' \mid N \in \mathcal{N}_\sigma, N' \in \mathcal{N}_\tau\}$$



$\mathcal{N}_\sigma$



$\mathcal{N}_\sigma \wedge \mathcal{N}_\tau$



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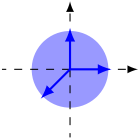

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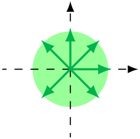
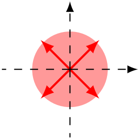


$\mathcal{R}$



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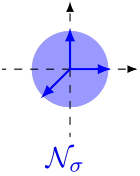


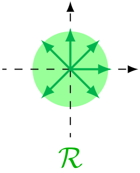


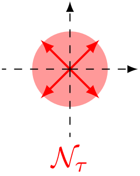


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# Exact formula of $V(x)$ for all $x$

There exists a collection $\mathcal{I}_\sigma$ of active constraints sets such that $\mathcal{N}_\sigma = \{\mathrm{Cone}(W_I^\top) \mid I \in \mathcal{I}_\sigma\}$.

## Theorem

*For all $I \in \mathcal{I}_\sigma$, there exists $\mu(I) \in \mathbb{R}_+^I$ such that*

$$-\mu(I)^\top W_I = \mathbb{E}\left[\mathbb{1}_{\mathbf{c} \in -\,\mathrm{ri}\,\mathrm{Cone}(W_I^\top)} \mathbf{c}^\top\right]$$

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*In particular, $V$ is polyhedral.*

Bonus: *for all distributions of* $\mathbf{c}$, $V$ is affine on each cell of $\mathcal{C}(P,\pi)$.

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# Extension to multistage and stochastic constraints

**Theorem**

*All results generalize to stochastic constraints with finite support and multistage*

- ⇝ *All $V_t$ are polyhedral* *(easy)*
- ⇝ *The regions where $(V_t)_t$ is affine do not depend on the $(\mathbf{c}_t)_t$* *(harder)*
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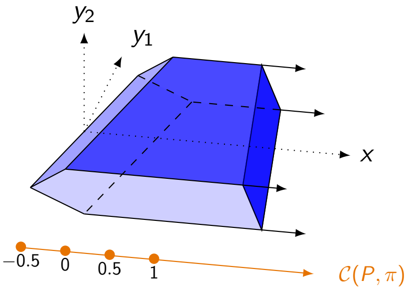
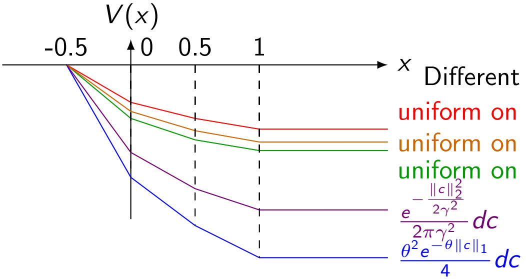
# Contents

# Explicit computation of the example

$$V(x) = \mathbb{E}\begin{bmatrix} \min\limits_{y\in\mathbb{R}^2} & \mathbf{c}^\top y \\ \text{s.t.} & \|y\|_1 \leqslant 1 \\ & y_1 \leqslant x \\ & y_2 \leqslant x \end{bmatrix}$$

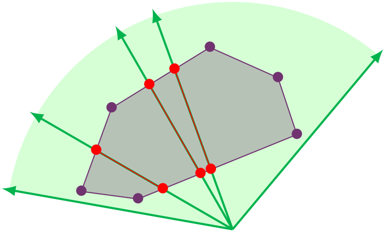


# We can triangulate $-N \cap \mathrm{supp}(\mathbf{c})$

We need to compute $\mathbb{E}\big[\mathbf{c}\mathbb{1}_{\mathbf{c}\in \mathrm{ri}(N)}\big] = \mathbb{P}\big[\mathbf{c} \in \mathrm{ri}(N)\big]\mathbb{E}\big[\mathbf{c} \mid \mathbf{c} \in \mathrm{ri}(N)\big]$ which is additive.

The shape of $-N \cap \mathrm{supp}(\mathbf{c})$ can be complicated.



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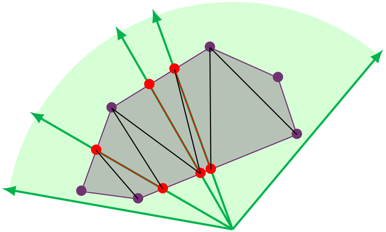

$\mathcal{N}(P_x) \cap -\text{supp}(\mathbf{c})$

# Explicit formulas for usual distributions

We can compute explicitly $\mathbb{E}\left[\mathbf{c}\mathbb{1}_{\mathbf{c}\in \mathrm{ri}(N)}\right] = \mathbb{P}\left[\mathbf{c} \in \mathrm{ri}(N)\right]\mathbb{E}\left[\mathbf{c} \mid \mathbf{c} \in \mathrm{ri}(N)\right]$ for classical distributions

| Distribution | Uniform on polytope | Exponential | Gaussian |
|---|---|---|---|
| $d\mathbb{P}(c)$ | $\frac{\mathbb{1}_{c\in Q}}{\mathrm{Vol}_d(Q)} d\mathcal{L}_{\mathrm{Aff}(Q)}(c)$ | $\frac{e^{\theta^\top c}\mathbb{1}_{c\in K}}{\Phi_K(\theta)} d\mathcal{L}_{\mathrm{Aff}(K)} c$ | $\frac{e^{-\frac{1}{2}c^\top M^{-2}c}}{(2\pi)^{\frac{m}{2}} \det M} dc$ |
| Support | Polytope : $Q$ | Cone : $K$ | $\mathbb{R}^m$ |
| $\mathbb{P}\left[\mathbf{c} \in S\right]$ | $\frac{\mathrm{Vol}_d(S)}{\mathrm{Vol}_d(Q)}$ | $\frac{\lvert \det(\mathrm{Ray}(S))\rvert}{\Phi_K(\theta)} \prod_{r\in \mathrm{Ray}(S)} \frac{1}{-r^\top \theta}$ | $\mathrm{Ang}\left(M^{-1}S\right)$ |
| $\mathbb{E}\left[\mathbf{c} \mid \mathbf{c} \in S\right]$ | $\frac{1}{d}\sum_{v\in \mathrm{Vert}(S)} v$ | $\left(\sum_{r\in \mathrm{Ray}(S)} \frac{-r_i}{r^\top \theta}\right)_{i\in[m]}$ | $\frac{\sqrt{2}\Gamma(\frac{m+1}{2})}{\Gamma(\frac{m}{2})} M\,\mathrm{Centr}\left(S\cap \mathbb{S}_{m-1}\right)$ |

These formulas are valid for $S$ full dimensional simplex or simplicial cone.

# Contents

# Earlier and new complexity results

Volume of a polytope

$$\mathrm{Vol}\left(\{z \in \mathbb{R}^d \mid Az \leqslant b\}\right) \text{ or}$$
$$\mathrm{Vol}\left(\mathrm{Conv}(v_1, \cdots, v_n)\right)$$

- $\sharp P$-complete: Dyer and Frieze (1988)
- Polynomial for fixed dimension $d$: Barvinok (1994)

2-stage linear problem

$$\min_{x \in \mathbb{R}^n} c_0^\top x + \mathbb{I}_{Ax \leqslant b} + \mathbb{E}\Big[\min_{y \in \mathbb{R}^m} \mathbf{c}^\top y + \mathbb{I}_{\mathbf{T}x + \mathbf{W}y \leqslant \mathbf{h}}\Big]$$

- $\sharp P$-hard: Hanasusanto, Kuhn and Wiesemann (2016)
- Polynomial for fixed $m$ ?

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- $\sharp P$-hard: Hanasusanto, Kuhn and Wiesemann (2016)
- Polynomial for fixed $m$ for some usual distributions: FGL (2020)

# Complexity results

We make the following assumption:
$\sharp\,\text{supp}(\mathbf{T}, \mathbf{W}, \mathbf{h})$ is finite and $\mathbb{P}\big[\mathbf{c} \in S\big]$ and $\mathbb{E}\big[\mathbf{c} \,|\, \mathbf{c} \in S\big]$ can be computed in polynomial time for S full dimensional simplex or simplicial fan.

**Theorem**

$$\min_{x \in \mathbb{R}^n} \quad c_0^\top x + \mathbb{I}_{Ax \leqslant b} + \mathbb{E}\Big[\min_{y \in \mathbb{R}^m} \mathbf{c}^\top y + \mathbb{I}_{\mathbf{T}x + \mathbf{W}y \leqslant \mathbf{h}}\Big] \tag{2SLP}$$

*When m is fixed, we can solve (2SLP) in polynomial time.*

**Theorem**

*For $x \in \mathbb{R}^n$*

$$V(x) := \mathbb{E}\Big[\min_{y \in \mathbb{R}^m} \mathbf{c}^\top y + \mathbb{I}_{\mathbf{T}x + \mathbf{W}y \leqslant \mathbf{h}}\Big]$$

*When n, m and $\sharp\,\text{supp}(\mathbf{T}, \mathbf{W}, \mathbf{h})$ are fixed, we can compute the epigraph of V in polynomial time.*

# Complexity result multistage

We can generalize to multistage by fixing several dimensions and the horizon.

**Theorem (MSLP is polynomial for fixed dimensions)**

*Assume that* $t_{\max} \geqslant 3, n_2, \dots, n_{t_{\max}}, \sharp\big(\mathrm{supp}(\mathbf{T}_2, \mathbf{W}_2, \mathbf{h}_2)\big),$ $\cdots, \sharp\big(\mathrm{supp}(\mathbf{T}_{t_{\max}}, \mathbf{W}_{t_{\max}}, \mathbf{h}_{t_{\max}})\big)$ *are fixed integers*

*and for all* $t \in [t_{\max}]$, $\mathbf{c}_t$ *conditionally to* $\{(\mathbf{T}_t, \mathbf{W}_t, \mathbf{h}_t) = (T, W, h)\}$ *is easily computable.*

*Then, we can solve MSLP in polynomial time.*

# Conclusion

- $V_t$ are polyhedral
  and affine on regions that are independent of the cost distribution;
- MSLP with arbitrary cost distribution can be exactly discretized;
- analytical formulas for some usual distributions;
- fixed-parameter versions of 2SLP and MSLP are polynomial time.

Perspectives

$\rightsquigarrow$ Deduce a faster algorithm from the algebraic structure

$\rightsquigarrow$ Extend the complexity analysis to approximation of the problem;

$\rightsquigarrow$ Extend to integer stochastic problems;

$\rightsquigarrow$ Distributionnally robust optimization or sensibility analysis.

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# Thank you for listening ! Any question ?

- Maël Forcier and Stéphane Gaubert and Vincent Leclère, *The Polyhedral Structure and Complexity of Multistage Stochastic Linear Problem with General Cost Distribution*, EasyChair Preprint no. 4113, 2020. `https://easychair.org/publications/preprint/pFGX`.