Generalized adaptive partition based method for 2 stage stochastic linear problems

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ROADEF

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2 stage stochastic linear programming (2SLP)

$$\begin{aligned} \min_{x \in \mathbb{R}^n_+} \quad & c^\top x + \mathbb{E}\big[Q(x, \boldsymbol{\xi})\big] \\ \text{s.t.} \quad & Ax = b \end{aligned}$$

where $\boldsymbol{\xi} = (\boldsymbol{T}, \boldsymbol{h})$ is random whereas q and W are deterministic[1]

$$Q(x, \xi) := \min_{y \in \mathbb{R}^m_+} q^\top y \qquad = \max_{\lambda \in \mathbb{R}^n} (h - Tx)^\top \lambda$$

$$\text{s.t.} \quad Tx + Wy = h \qquad \text{s.t.} \quad W^\top \lambda \leqslant q$$

We define

$$X := \{x \in \mathbb{R}^n_+ \mid Ax = b\} \qquad D := \{\lambda \in \mathbb{R}^l \mid W^\top \lambda \leqslant q\}$$

[1] Can be extended to generic random $\boldsymbol{q}$, and finitely supported $\boldsymbol{W}$

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$\rightsquigarrow$ need to discretize $\boldsymbol{\xi}$

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Sample Average Approximation

$$\min_{x \in X} c^\top x + V(x) \quad \text{where} \quad V(x) := \mathbb{E}\big[Q(x, \boldsymbol{\xi})\big] \tag{2SLP}$$

Randomly draw $\xi^1, \cdots, \xi^N$ and consider

$$\min_{x \in X} c^\top x + V_N^{SAA}(x) \quad \text{where} \quad V_N^{SAA}(x) := \frac{1}{N} \sum_{k=1}^{N} Q(x, \xi^k) \tag{$2SLP_N$}$$

Solve the equivalent finite LP

$$\begin{aligned} \min_{x \in X, (y_k)_{k=1}^N \in (\mathbb{R}_+^m)^N} \quad & c^\top x + \frac{1}{N} \sum_{k=1}^{N} q^\top y_k \\ & T^k x + W y_k \leqslant h^k \qquad \forall k = 1..N \end{aligned} \tag{$2SLP_N$}$$

By statistical results, $Val(2SLP_N) \to_{N \to \infty} Val(2SLP)$.

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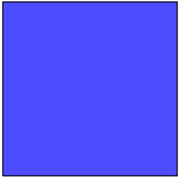
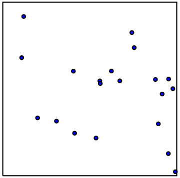
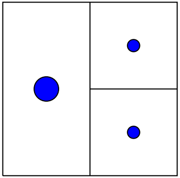
Partitioning the cost-to-go function



ξ continuous



SAA $N = 20$



Partition

$$V(x) = \mathbb{E}\big[Q\big(x, \boldsymbol{\xi}\big)\big]$$

$$V_N^{SAA}(x) = \frac{1}{N} \sum_{k=1}^{N} Q\big(x, \xi^k\big)$$

$$V_{\mathcal{P}}(x)$$

Definition (Expected-cost-go of partition)

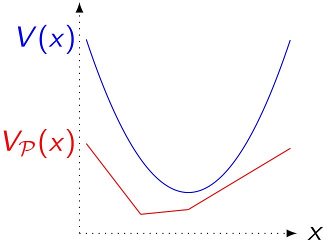
Let $\mathcal{P}$ be a $\mathbb{P}$-partition of Ξ, we define

$$V_{\mathcal{P}}(x) := \sum_{P \in \mathcal{P}} \mathbb{P}\big[P\big] Q\big(x, \mathbb{E}\big[\boldsymbol{\xi} | P\big]\big)$$

Property of cost-to-go partition

$$V_{\mathcal{P}}(x) := \sum_{P \in \mathcal{P}} \mathbb{P}\big[P\big]\, Q\Big(x, \mathbb{E}\big[\boldsymbol{\xi} \big| P\big]\Big)$$

For all x, $Q(x, \cdot)$ is convex,
then $V_{\mathcal{P}} \leqslant V$
For all P, $Q\Big(\cdot, \mathbb{E}\big[\boldsymbol{\xi} \big| P\big]\Big)$ is polyhedral
thus $V_{\mathcal{P}}$ is polyhedral.



The $(2SLP_{\mathcal{P}})$ problem $\min_{x \in X} c^\top x + V_{\mathcal{P}}(x)$ is the equivalent finite LP

$$\min_{x \in X, (y_P)_{P \in \mathcal{P}} \in (\mathbb{R}^m_+)^{\mathcal{P}}} \quad c^\top x + \sum_{P \in \mathcal{P}} \mathbb{P}\big[P\big] q^\top y_P \qquad (2SLP_{\mathcal{P}})$$

$$\mathbb{E}\big[\boldsymbol{T} \big| P\big] x + W y_P \leqslant \mathbb{E}\big[\boldsymbol{h} \big| P\big] \qquad \forall P \in \mathcal{P}$$

Property of cost-to-go partition

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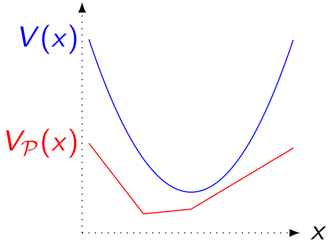

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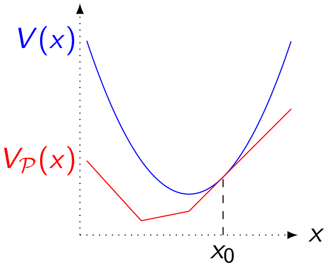
$$\mathbb{E}\big[\boldsymbol{T}|P\big] x + W y_P \leqslant \mathbb{E}\big[\boldsymbol{h}|P\big] \qquad \forall P \in \mathcal{P}$$

Adapted partition

Definition

We say that a partition $\mathcal{P}$ is adapted to x_0 if

$$V_{\mathcal{P}}(x_0) = V(x_0) := \mathbb{E}\big[Q(x_0, \boldsymbol{\xi})\big]$$



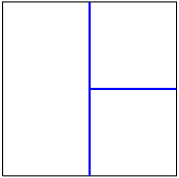
Refinement

We say that $\mathcal{R}$ refines $\mathcal{P}$ and we denote $\mathcal{R} \preccurlyeq \mathcal{P}$ if

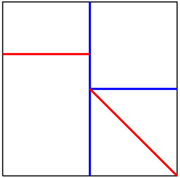
$$\forall R \in \mathcal{R}, \exists P \in P, R \subset P$$

We denote $\preccurlyeq_{\mathbb{P}}$ the refinement relation $\mathcal{R}$ up to $\mathbb{P}$-negligeable sets. Then,

$$\mathcal{R} \preccurlyeq_{\mathbb{P}} \mathcal{P} \Rightarrow V_{\mathcal{P}} \leqslant V_{\mathcal{R}}$$



$\mathcal{P}$



$\mathcal{R}$

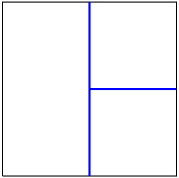
Common Refinement

We define $\mathcal{P} \preccurlyeq \mathcal{P}'$ the common refinement of $\mathcal{P}$ and $\mathcal{P}'$

$$\mathcal{P} \wedge \mathcal{P}' = \{P \cap P' \mid P \in \mathcal{P}, P' \in \mathcal{P}'\}$$

Since $\mathcal{P} \wedge \mathcal{P}'$ refines $\mathcal{P}$ and $\mathcal{P}'$

$$\max(V_{\mathcal{P}}, V_{\mathcal{P}'}) \leqslant V_{\mathcal{P} \wedge \mathcal{P}'}$$



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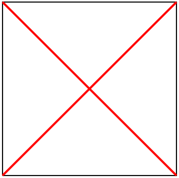
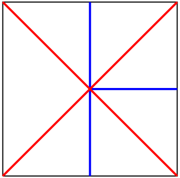


$\mathcal{P}'$



$\mathcal{P} \wedge \mathcal{P}'$

General framework for APM

Algorithm General framework for APM methods

$k \leftarrow 0$, $z_0^U \leftarrow +\infty$, $z_0^L \leftarrow -\infty$, $\mathcal{P}^0 \leftarrow \{\Xi\}$
while $z_k^U - z_k^L > \varepsilon$ **do**
    Solve $z_k^L \leftarrow \min_{x \in X} c^\top x + V_{\mathcal{P}^{k-1}}(x)$ and let x_k be an optimal solution i.e. solve a finite (2SLP)
    Choose a partition $\mathcal{P}_{x_k}$ adapted to x_k
    $\mathcal{P}^k \leftarrow \mathcal{P}^{k-1} \wedge \mathcal{P}_{x_k}$
    for $P \in \mathcal{P}^k$ **do**
        Compute $\mathbb{P}\big[P\big]$ and $\mathbb{E}\big[\boldsymbol{\xi}|P\big]$
    end for
    $z_k^U \leftarrow \min\Big(z_{k-1}^U, c^\top x_k + V_{\mathcal{P}^k}(x_k)\Big)$
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Finite case - Song and Luedtke

Song and Luedtke APM algorithm apply to 2SLP with finitely supported random variable.

Lemma

Let $\mathcal{P}$ a partition of Ξ. $\mathcal{P}$ is adapted at x iff for all set of scenarios $P \in \mathcal{P}$, there exists a common optimal multiplier λ_P, i.e.

$$\forall P \in \mathcal{P}, \exists \lambda_P \in D, \forall \xi_k \in P, \lambda_P \in \operatorname*{argmax}_{\lambda \in D} (h^k - T^k x)^\top \lambda$$

Idea

- Sample a large number of scenario
- without loss of precision
- gather the scenarios thanks to this condition

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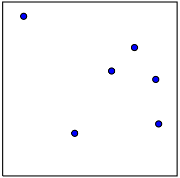

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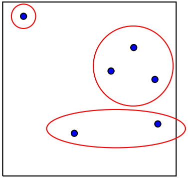

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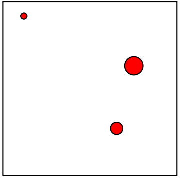

Ramirez-Pico and Moreno GAPM

Idea : Partition directly Ξ instead of sampling first

Lemma (Ramirez-Pico Moreno)

Let $\mathcal{P}$ a partition of Ξ. If there exists an optimal $\lambda(\boldsymbol{\xi})$ such that, for all $P \in \mathcal{P}$,

$$\mathbb{E}\big[\boldsymbol{h}|P\big]^\top \mathbb{E}\big[\lambda(\boldsymbol{\xi})|P\big] = \mathbb{E}\big[\boldsymbol{h}^\top \lambda(\boldsymbol{\xi})|P\big]$$
$$x^\top \mathbb{E}\big[\boldsymbol{T}|P\big]^\top \mathbb{E}\big[\lambda(\boldsymbol{\xi})|P\big] = x^\top \mathbb{E}\big[\boldsymbol{T}^\top \lambda(\boldsymbol{\xi})|P\big]$$

then $\mathcal{P}$ is an adapted partition.

Unfortunately, we do not know an explicit algorithm to find a partition that satisfies this condition.

Comparison between partition based method

	APM	GAPM	G^2APM
Paper	Song, Luedtke (2015)	Ramirez-Pico, Moreno (2020)	F., Leclère (2021)
Non-finite supp(ξ)	×	✓	✓
Proof of convergence	✓	×	✓
Explicit formulation	✓	×	✓
Complexity result	×	×	✓
Fast iteration	✓	×	×

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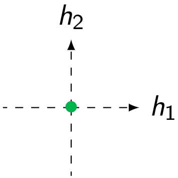
Normal fan $\mathcal{N}(D)$

Definition

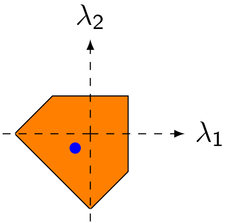
The normal fan of the polyhedron D is

$$\mathcal{N}(D) := \{N_D(\lambda) \mid \lambda \in D\}$$

with $N_D(\lambda) = \{h \mid \forall \lambda' \in P,\ h^\top(\lambda' - \lambda) \leqslant 0\}$ *the normal cone of* D *on* λ.



$N_D(\lambda)$



D λ and $N_D(\lambda)$

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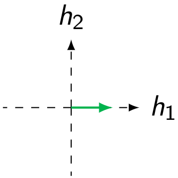
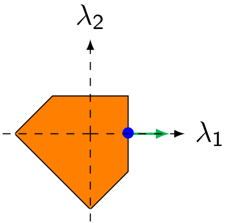


$N_D(\lambda)$



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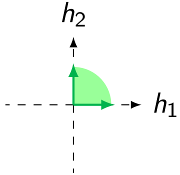

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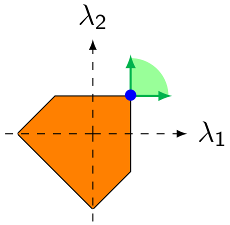

D λ and $N_D(\lambda)$

Normal fan $\mathcal{N}(D)$

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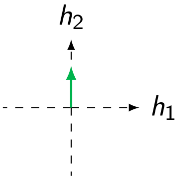

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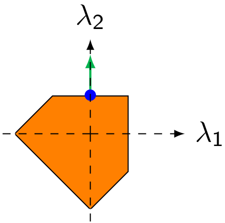

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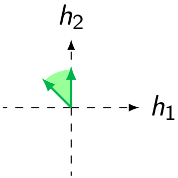

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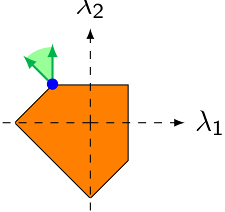

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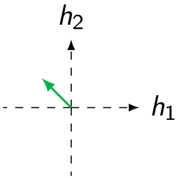

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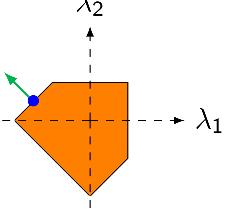

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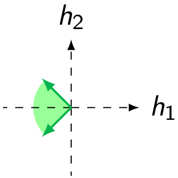

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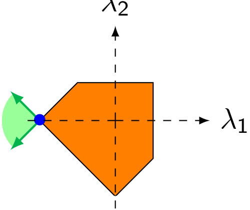

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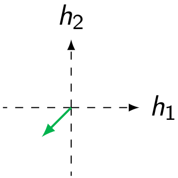

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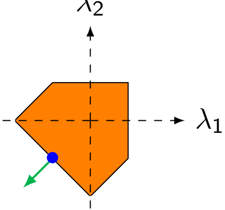

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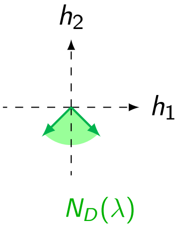

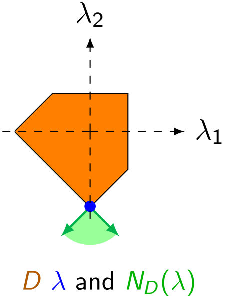

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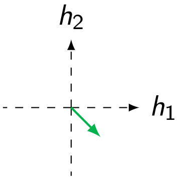

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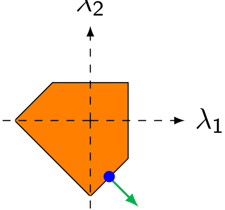

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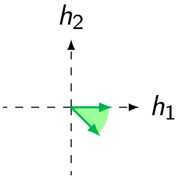

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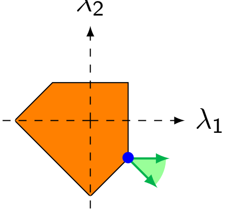

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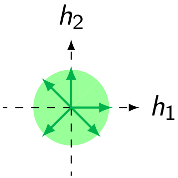
Proposition

$\{\mathrm{ri}(N) \mid N \in \mathcal{N}(D)\}$ *is a partition of* $\mathrm{supp}\,\mathcal{N}(D)$ *($= \mathbb{R}^m$ if D is bounded).*



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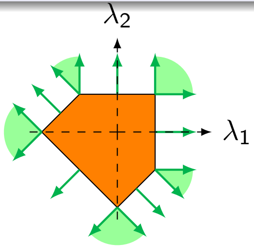

D and $\mathcal{N}(D)$

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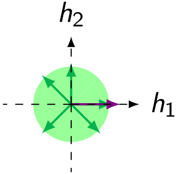

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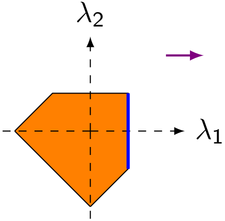

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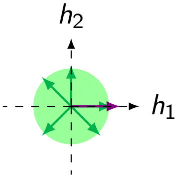

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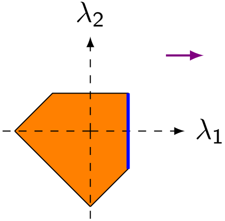

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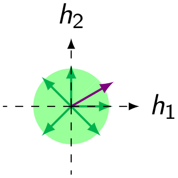

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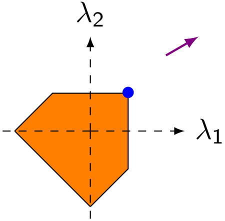

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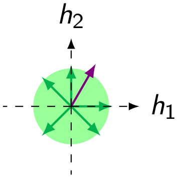

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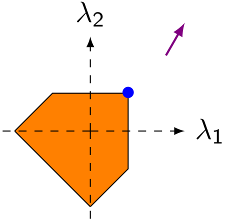

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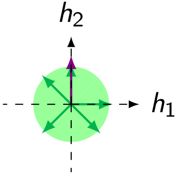

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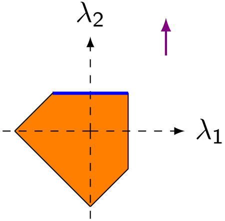

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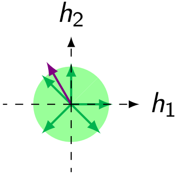

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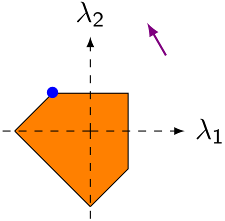

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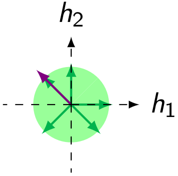

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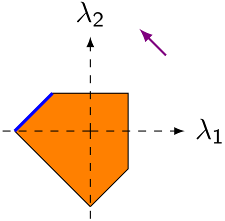

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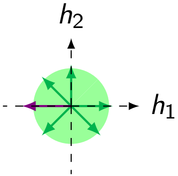

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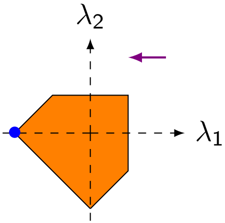

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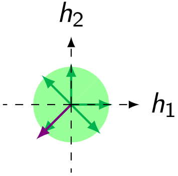

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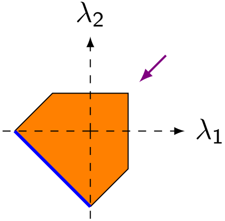

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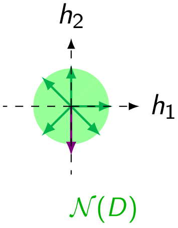

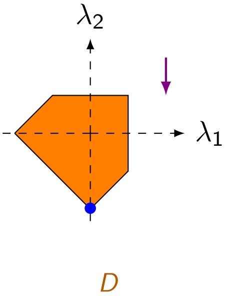

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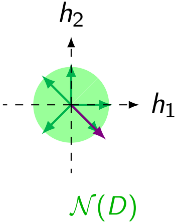

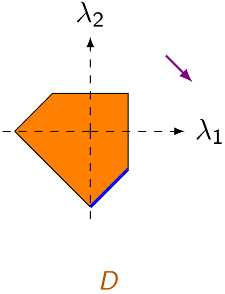

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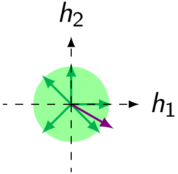

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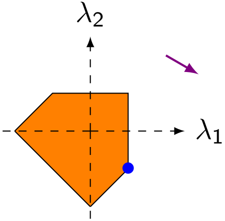

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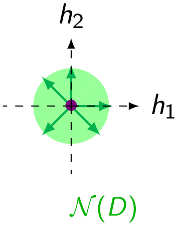

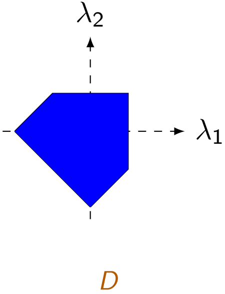

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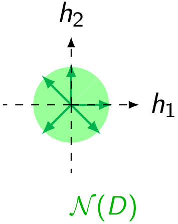

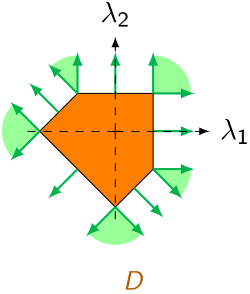

In particular, there exists a common optimal multipler λ_N for all $h - Tx \in \operatorname{ri} N$, *i.e.* where $Q(x, \xi) = (h - Tx)^\top \lambda_N$.

Contents

An explicit adapted partition

Consider $x \in \mathbb{R}^n$ and $N \in \mathcal{N}(\mathcal{D})$ a normal cone of D. We define

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$\rightsquigarrow$ Is it the coarsest one ?

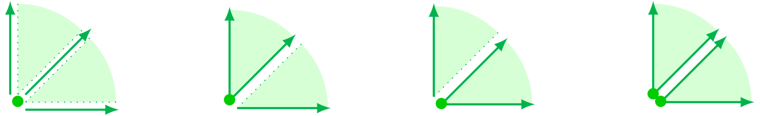
CNS conditions for a partition to be adapted

Theorem (FL 2021)

Consider $x \in \mathbb{R}^n$ and $\mathcal{P}$ a partition of Ξ. Then, there exists a canonical cover $\overline{\mathcal{R}}_x$ of Ξ (not necessarily a partition), is such that

$$\mathcal{P} \preccurlyeq_{\mathbb{P}} \mathcal{R}_x \implies V_{\mathcal{P}}(x) = V(x)$$
$$\mathcal{P} \preccurlyeq_{\mathbb{P}} \overline{\mathcal{R}}_x \iff V_{\mathcal{P}}(x) = V(x).$$

If $\boldsymbol{\xi}$ admits a density, $\mathcal{R}_x =_{\mathbb{P}} \overline{\mathcal{R}}_x$.



$\mathcal{R}_x$ $\qquad$ $\mathcal{P}$ $\qquad$ $\mathcal{P}'$ $\qquad$ $\overline{\mathcal{R}}_x$

$$E_{N,x} := \{\xi \in \Xi \mid h - Tx \in \mathrm{ri}(N)\}$$
$$\mathcal{R}_x := \big\{E_{N,x} \mid N \in \mathcal{N}(D)\big\}$$

$$\overline{E}_{N,x} := \{\xi \in \Xi \mid h - Tx \in N\}$$
$$\overline{\mathcal{R}}_x := \big\{E_{N,x} \mid N \in \mathcal{N}(D)^{\max}\big\}.$$

Subgradient of partition function

Recall that if $\mathcal{P} \preccurlyeq_{\mathbb{P}} \mathcal{R}_x$ then

$$V_{\mathcal{R}_x}(x) = V_{\mathcal{P}}(x) = V(x)$$

$$V_{\mathcal{R}_x}(\cdot) \leqslant V_{\mathcal{P}}(\cdot) \leqslant V(\cdot)$$

Lemma

Let $x \in \mathrm{dom}(V)$ *and* $\mathcal{P}$ *be a refinement of* $\mathcal{R}_x$*, i.e.* $\mathcal{P} \preccurlyeq \mathcal{R}_x$*, then*

$$\partial V_{\mathcal{R}_x}(x) \subset \partial V_{\mathcal{P}}(x) \subset \partial V(x)$$

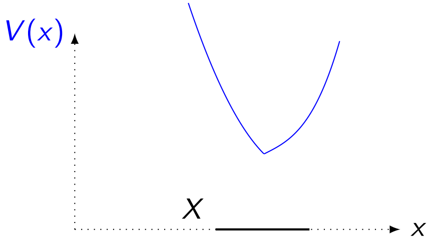
Furthermore, if $x \in \mathrm{ri}\,\mathrm{dom}(V)$,

$$\partial V_{\mathcal{R}_x}(x) = \partial V_{\mathcal{P}}(x) = \partial V(x)$$

Contents

Link with Benders decomposition and L-shaped

Partition based method can be seen as a tangent cone method: a cutting plane method where we add all active cuts instead of a single one.



Theorem (Convergence and complexity results)

If $X \subset \mathbb{R}^+$ is contained in a ball of diameter $D \in \mathbb{R}^+$ and $x \to c^\top x + V(x)$ is Lipschitz with constant L then the partition based method finds an ε-solution in at most $\left(\frac{LD}{\varepsilon} + 1\right)^n$ iterations.

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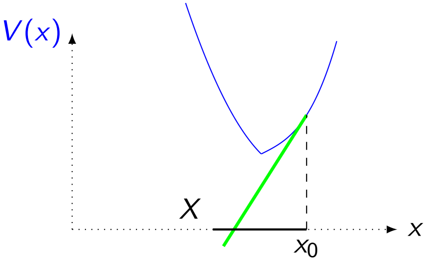

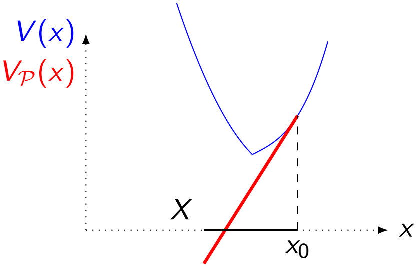

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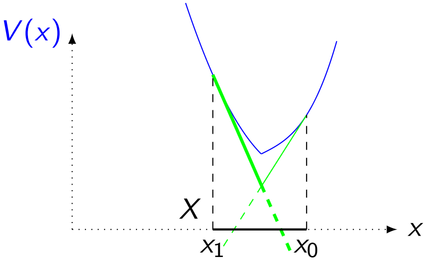

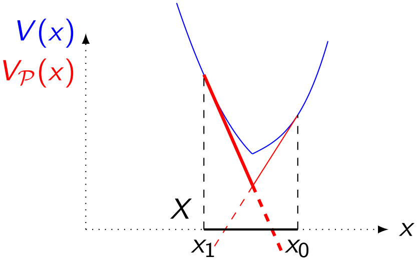

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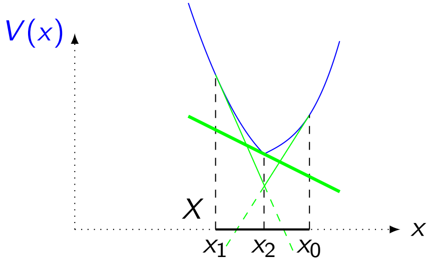

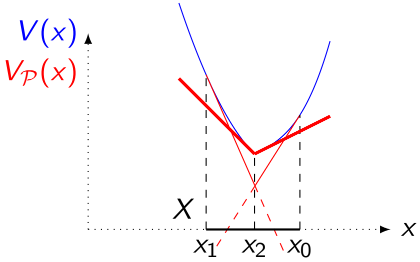

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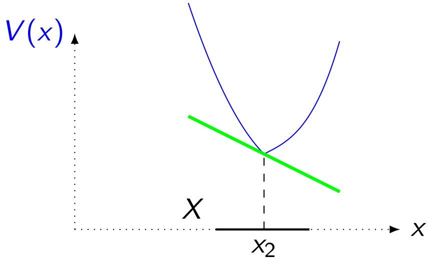

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Contents

Numerical Results - ProdMix

k	z_L^k	z_U^k	$z_U^k - z_L^k$	Total time	$\|\mathcal{P}^k\|$
1	-18666.67	-16939.71	1726.96	0.57 s	4
2	-17873.01	-17383.73	489.28	2.1 s	9
4	-17744.67	-17709.00	35.67	9.1 s	25
6	-17713.74	-17711.37	2.37	23.7 s	49
8	-17711.71	-17711.56	0.15	50.0 s	81
10	-17711.57	-17711.56	0.01	88.0 s	121

Table: Results for problem Prod-Mix

Comparison with SAA : we solved the same problem 100 times, each with 10 000 scenarios randomly drawn

$\rightsquigarrow$ 95% confidence interval centered in -17711, with radius 2.2.

$\rightsquigarrow$ required 2058s of computation.

Perspectives

A GAPM iteration is very slow in high dimension

$\rightsquigarrow$ Compute $\mathbb{E}\left[\boldsymbol{\xi}|N\right]$ and $\mathbb{P}\left[N\right]$ with approximations and compare with SAA

The size of the partition can grow quickly

$\rightsquigarrow$ Find some heuristics for not only refining but merging which is equivalent to forget cuts for cutting planes method.

$\rightsquigarrow$ Implement with bundle methods.

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References

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Explicit representation of $E_{N,x}$

Let $N := \{\widetilde{h} \,|\, M\widetilde{h} \leqslant 0\}$
Then

$$
\begin{aligned}
E_{N,x} &= \{\xi \in \Xi \,|\, h - Tx \in \mathrm{ri}(N)\} \\
&= \{\xi \in \Xi \,|\, M(h - Tx) < 0\} \\
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where $H^x = (-x_1 M \cdots - x_n M \; M)$.

If $\boldsymbol{T} \equiv T$ is deterministic,

$$\mathcal{R}_x = Tx + \mathcal{N}(D)$$

Then, we only need to compute $\mathcal{N}(D)$ once and translate at each iteration.

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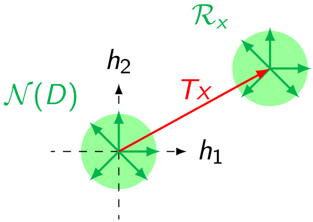

Explicit formulas for usual distributions

Recall that $V_{\mathcal{P}}(x) = \sum_{P\in\mathcal{P}} \mathbb{P}\big[P\big] Q\big(x, \mathbb{E}\big[\boldsymbol{\xi}|P\big]\big)$.

Thus, we need to compute $\mathbb{P}\big[C\big]$ and $\mathbb{E}\big[\boldsymbol{\xi} \mid C\big]$ when C is a polyhedron.

Fortunately we have some explicit formulas, valid for S full dimensional simplex or simplicial cone, which can be used through triangulation.

Distribution	Uniform on polytope	Exponential	Gaussian
$d\mathbb{P}(\xi)$	$\frac{\mathbb{1}_{\xi\in Q}}{\mathrm{Vol}_d(Q)} \mathcal{L}_{\mathrm{Aff}(Q)}(d\xi)$	$\frac{e^{\theta^\top \xi} \mathbb{1}_{\xi\in K}}{\Phi_K(\theta)} \mathcal{L}_{\mathrm{Aff}(K)}(d\xi)$	$\frac{e^{-\frac{1}{2}\xi^\top M^{-2}\xi}}{(2\pi)^{\frac{m}{2}} \det M} d\xi$
Support	Polytope : Q	Cone : K	$\mathbb{R}^m$
$\mathbb{P}\big[S\big]$	$\dfrac{\mathrm{Vol}_d(S)}{\mathrm{Vol}_d(Q)}$	$\dfrac{\lvert\det(\mathrm{Ray}(S))\rvert}{\Phi_K(\theta)} \prod_{r\in\mathrm{Ray}(S)} \dfrac{1}{-r^\top\theta}$	$\mathrm{Ang}\left(M^{-1}S\right)$
$\mathbb{E}\big[\xi \mid S\big]$	$\frac{1}{d}\sum_{v\in\mathrm{Vert}(S)} v$	$\left(\sum_{r\in\mathrm{Ray}(S)} \frac{-r_i}{r^\top\theta}\right)_{i\in[m]}$	$\frac{\sqrt{2}\Gamma(\frac{m+1}{2})}{\Gamma(\frac{m}{2})} M \,\mathrm{Ctr}\left(S\cap\mathbb{S}_{m-1}\right)$

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