Multistage Stochastic Linear Problem and Polyhedral Geometry

Maël Forcier, Stéphane Gaubert, Vincent Leclère

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mael.forcier@enpc.fr

Multistage stochastic linear programming (MSLP)

$$\begin{aligned}
\min_{(\mathbf{x}_t)_{t\in[T]}} \quad & \mathbb{E}\Big[\sum_{t=1}^{T} \mathbf{c}_t^\top \mathbf{x}_t\Big] & \\
\text{s.t.} \quad & \mathbf{A}_t \mathbf{x}_t + \mathbf{B}_t \mathbf{x}_{t-1} \leqslant \mathbf{b}_t & \forall t \in [T] \\
& \mathbf{x}_t \text{ random variable in } \mathbb{R}^{n_t} & \forall t \in [T] \\
& \sigma(\mathbf{x}_t) \subset \sigma(\mathbf{c}_k, \mathbf{A}_k, \mathbf{B}_k, \mathbf{b}_k)_{k \leqslant t} & \forall t \in [T] \\
& \mathbf{x}_0 \equiv x_0 \text{ given} &
\end{aligned}$$

where $\mathbf{c}_t \in \mathbb{R}^{n_t}$, $\mathbf{A}_t \in \mathbb{R}^{q_t \times n_{t-1}}$,$\mathbf{B}_t \in \mathbb{R}^{q_t \times n_t}$ and $\mathbf{b}_t \in \mathbb{R}^{q_t}$ are given random variables.
$(\mathbf{c}_t, \mathbf{A}_t, \mathbf{B}_t, \mathbf{b}_t)_{t\in[T]}$ is an independent sequence.

We set $V_{T+1} \equiv 0$ and:

$$V_t(x_{t-1}) := \mathbb{E}\left[\begin{array}{ll} \min\limits_{x_t \in \mathbb{R}^{n_t}} & \mathbf{c}_t^\top x_t + V_{t+1}(x_t) \\ \text{s.t.} & \mathbf{A}_t x_t + \mathbf{B}_t x_{t-1} \leqslant \mathbf{b}_t \end{array}\right]$$

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Quantization of a MSLP

The random variable $(\mathbf{c}_t, \mathbf{A}_t, \mathbf{B}_t, \mathbf{b}_t)_{t \in [T]}$ are often replaced by a discrete distribution on a finite number of scenarios

$$V_t(x_{t-1}) \simeq \widetilde{V}_t(x_{t-1}) = \sum_{k=1}^{K} p_k \begin{array}{rl} \min\limits_{x_t \in \mathbb{R}^{n_t}} & c_{t,k}^\top x_t + V_{t+1}(x_t) \\ \text{s.t.} & A_{t,k} x_t + B_{t,k} x_{t-1} \leqslant b_{t,k} \end{array}$$

Scenario drawn by Monte Carlo : Sample Average Approximation

Definition

We say that an MSLP admits an *exact quantization* if there exists a finitely supported $(\check{\mathbf{c}}_t, \check{\mathbf{A}}_t, \check{\mathbf{B}}_t, \check{\mathbf{b}}_t)_{t \in [T]}$ that yields the same expected cost-to-go functions, $(V_t)_{t \in [T]}$. In particular the MSLP is equivalent to a problem on a finite scenario tree.

Exact Quantization $\Rightarrow$ Polyhedral Functions

$$V(x) := \mathbb{E}\left[\begin{array}{ll}\min_{y \in \mathbb{R}^m} & \mathbf{c}^\top y \\ \text{s.t.} & \mathbf{A}x + \mathbf{B}y \leqslant \mathbf{b}\end{array}\right]$$

Theorem (see e.g. Shapiro, Dentcheva, Ruszczyński)

If $\mathbf{c}$, $\mathbf{A}$, $\mathbf{B}$, $\mathbf{b}$ *have a finite support, then* V *is polyhedral*

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Corollary

If there exists an exact quantization, then V *is polyhedral*

Counter examples with stochastic constraints

Stochastic left hand side constraint **B**

$$V(x) = \mathbb{E}\begin{bmatrix} \min\limits_{y\in\mathbb{R}^m} & y \\ \text{s.t.} & \mathbf{u}x \leqslant y \\ & 1 \leqslant y \end{bmatrix} = \mathbb{E}\big[\max(\mathbf{u}x, 1)\big] = \begin{cases} 1 & \text{if } x \leqslant 1 \\ \frac{x}{2} + \frac{1}{2x} & \text{if } x \geqslant 1 \end{cases}$$

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where **u** is uniform on $[0, 1]$.

$\rightsquigarrow$ Nevertheless, there exists an exact quantization when the cost **c** is stochastic !

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Illustrative running example:

$$P_x := \{y \in \mathbb{R}^m \mid \|y\|_1 \leqslant 1, \quad y_1 \leqslant x, \quad y_2 \leqslant x\}$$

P_x for $x = 0.8$

P_x for $x = 0.3$

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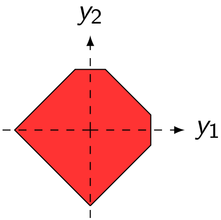

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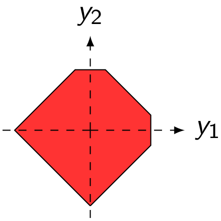

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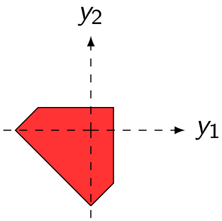


P_x for $x = 0.3$

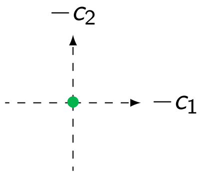
Normal fan $\mathcal{N}(P_x)$

Definition

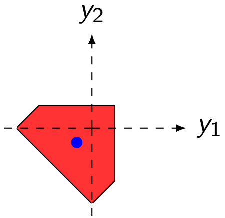
The normal fan of the fiber P_x *is*

$$\mathcal{N}(P_x) := \{N_{P_x}(y) \,|\, y \in P_x\}$$

with $N_{P_x}(y) = \{c \,|\, \forall y' \in P_x,\ c^\top(y' - y) \leqslant 0\}$ *the normal cone of* P_x *on* y*.*



$N_{P_x}(y)$ for $x = 0.3$



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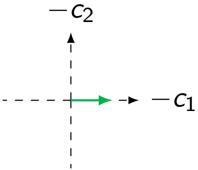

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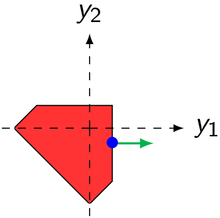

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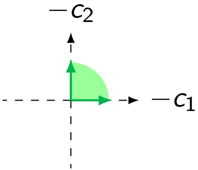

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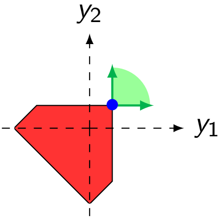

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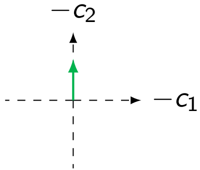

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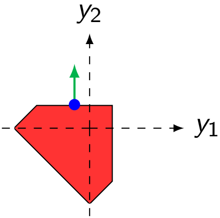

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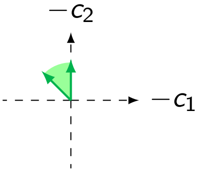

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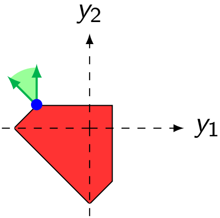

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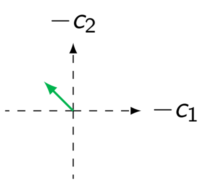

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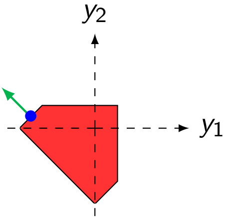

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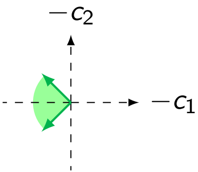

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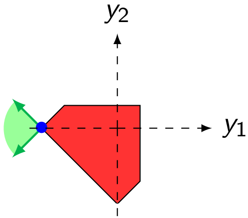

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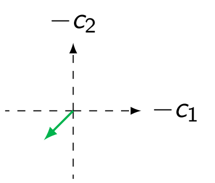

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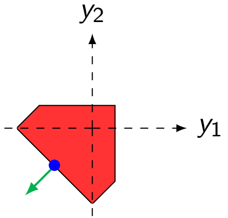

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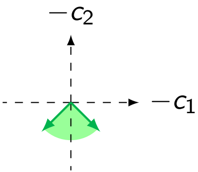

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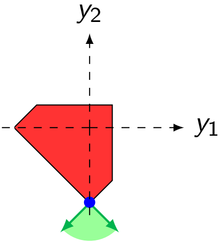

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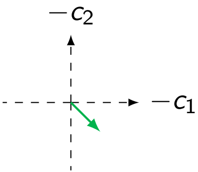

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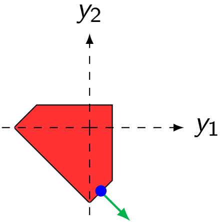

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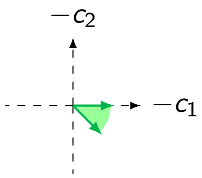

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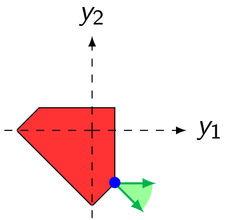

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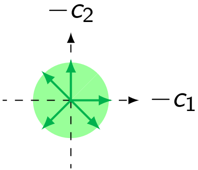
Proposition

If P_x *is bounded,* $\{\mathrm{ri}(N) \mid N \in \mathcal{N}(P_x)\}$ *is a partition of* $\mathbb{R}^m$*.*



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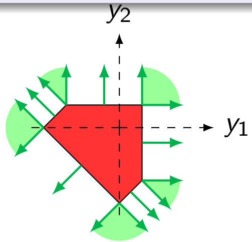

P_x and $\mathcal{N}(P_x)$ for $x = 0.3$

$\mathcal{N}(P_x)$: partition of cost coherent with the min

For a given x, we have

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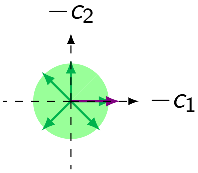
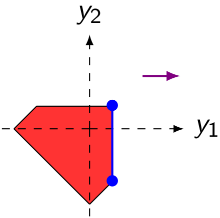


Cost $-c$ and $\mathcal{N}(P_x)$ for $x = 0.3$



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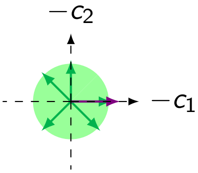

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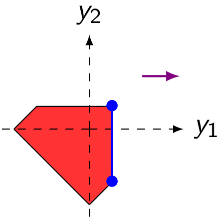

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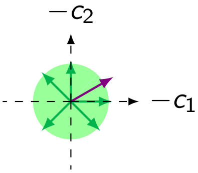

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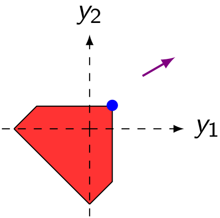

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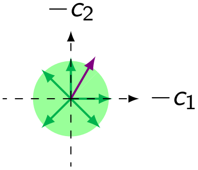

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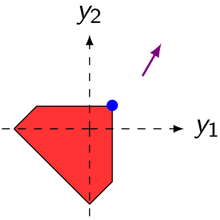

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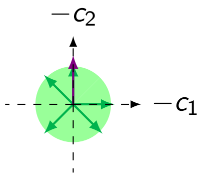

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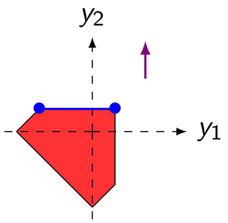

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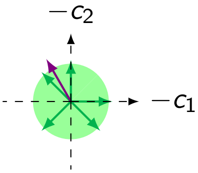

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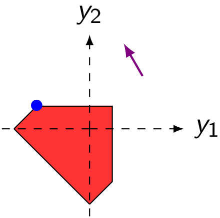

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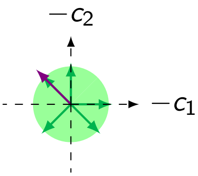

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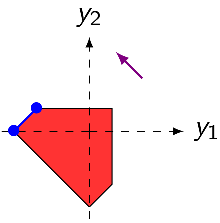

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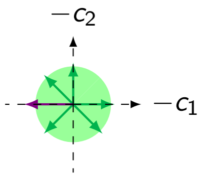

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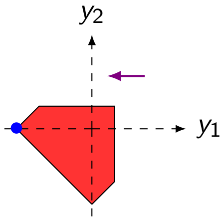

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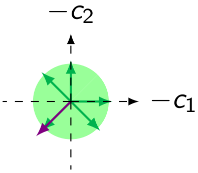

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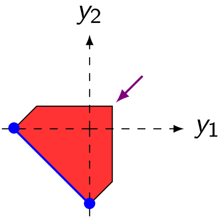

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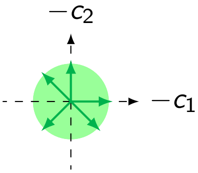

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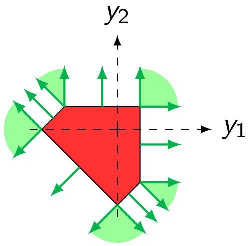


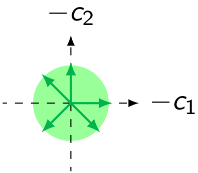
P_x for $x = 0.3$

Reduction to a finite sum

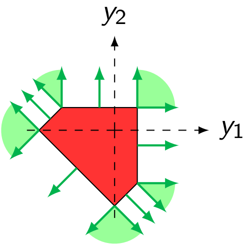
For a fixed x,

$$V(x) = \mathbb{E}\big[\min_{y \in P_x} \mathbf{c}^\top y\big] = \sum_{N \in \mathcal{N}(P_x)} \mathbb{E}\big[\mathbf{c}^\top \mathbb{1}_{\mathbf{c} \in -\operatorname{ri} N}\big] y_N(x)$$

where $y_N(x) \in \arg\min_{y \in P_x} c^\top y$ for any $c \in \operatorname{ri}(N)$.



$\mathcal{N}(P_x)$ for $x = 0.3$

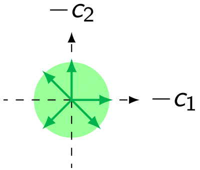


P_x and $\mathcal{N}(P_x)$ for $x = 0.3$

General cost $\mathbf{c}$ is equivalent to discrete cost $\check{\mathbf{c}}$ for given x

For a fixed x,

$$V(x) = \mathbb{E}\big[\min_{y \in P_x} \mathbf{c}^\top y\big]$$
$$= \sum_{N \in \mathcal{N}(P_x)} \mathbb{E}\big[\mathbf{c}^\top \mathbb{1}_{\mathbf{c} \in -\operatorname{ri} N}\big] y_N(x)$$



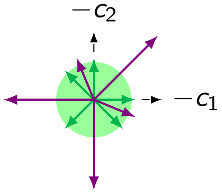
$\mathcal{N}(P_x)$ for $x = 0.3$

We draw a continuous cost $\mathbf{c}$.

General cost $\mathbf{c}$ is equivalent to discrete cost $\check{\mathbf{c}}$ for given x

For a fixed x,

$$\begin{aligned} V(x) &= \mathbb{E}\big[\min_{y \in P_x} \mathbf{c}^\top y\big] \\ &= \sum_{N \in \mathcal{N}(P_x)} \mathbb{E}\big[\mathbf{c}^\top \mathbb{1}_{\mathbf{c} \in -\operatorname{ri} N}\big] y_N(x) \\ &= \sum_{N \in \mathcal{N}(P_x)} p_N \check{c}_N{}^\top y_N(x) \end{aligned}$$



$\mathcal{N}(P_x)$ and $p_N \check{c}_N$ for $x = 0.3$

For $N \in \mathcal{N}(P_x)$,

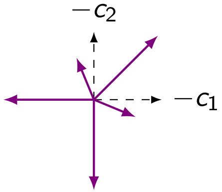
$$\begin{aligned} p_N &:= \mathbb{P}\big[\mathbf{c} \in -\operatorname{ri} N\big] \\ \check{c}_N &:= \mathbb{E}\big[\mathbf{c} | \mathbf{c} \in -\operatorname{ri} N\big] \end{aligned}$$

Instead of drawing a general $\mathbf{c}$, we draw a discrete cost $\check{\mathbf{c}}$ indexed by the finite collection $\mathcal{N}(P_x)$.

General cost $\mathbf{c}$ is equivalent to discrete cost $\check{\mathbf{c}}$ for given x

For a fixed x,

$$
\begin{aligned}
V(x) &= \mathbb{E}\Big[\min_{y \in P_x} \mathbf{c}^\top y\Big] \\
&= \sum_{N \in \mathcal{N}(P_x)} \mathbb{E}\big[\mathbf{c}^\top \mathbb{1}_{\mathbf{c} \in -\operatorname{ri} N}\big] y_N(x) \\
&= \sum_{N \in \mathcal{N}(P_x)} p_N \check{c}_N^\top y_N(x) \\
&= \sum_{N \in \mathcal{N}(P_x)} p_N \min_{y \in P_x} \check{c}_N^\top y
\end{aligned}
$$



$p_N \check{c}_N$ for $x = 0.3$

For $N \in \mathcal{N}(P_x)$,

$$
\begin{aligned}
p_N &:= \mathbb{P}\big[\mathbf{c} \in -\operatorname{ri} N\big] \\
\check{c}_N &:= \mathbb{E}\big[\mathbf{c} \,|\, \mathbf{c} \in -\operatorname{ri} N\big]
\end{aligned}
$$

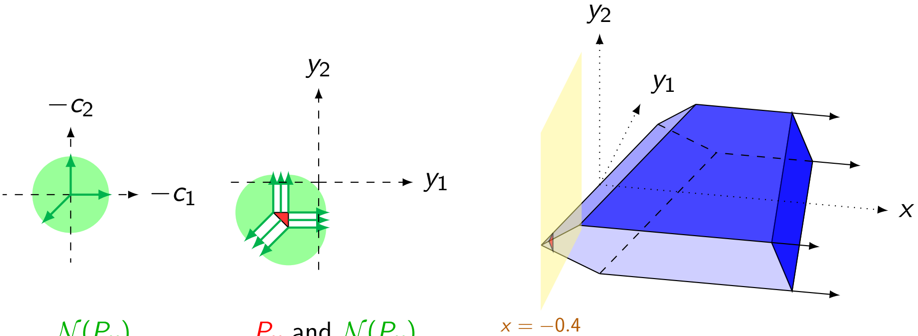
Instead of drawing a general $\mathbf{c}$, we draw a discrete cost $\check{\mathbf{c}}$ indexed by the finite collection $\mathcal{N}(P_x)$.

Contents

$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x, y) \mid Ax + By \leqslant b\} \quad \text{and} \quad P_x := \{y \mid Ax + By \leqslant b\}$$

$$x = -0.4$$



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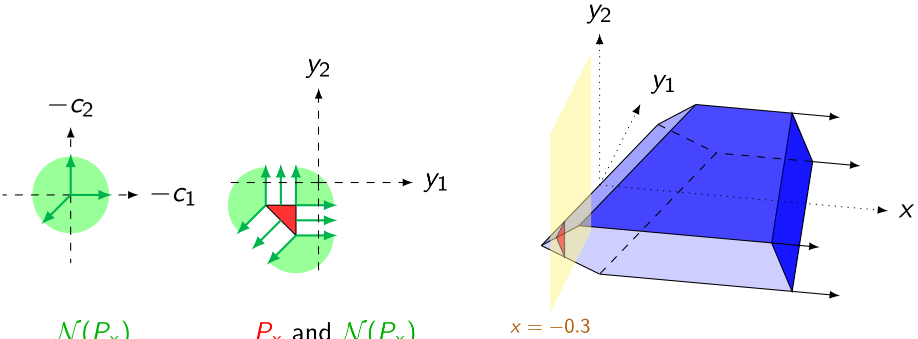

$\mathcal{N}(P_x)$ | P_x and $\mathcal{N}(P_x)$ | P and P_x

$\mathcal{N}(P_x)$ is piecewise constant with x.

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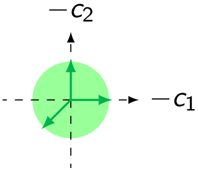
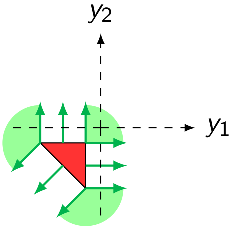
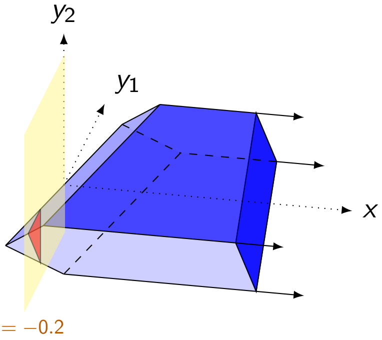


$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$



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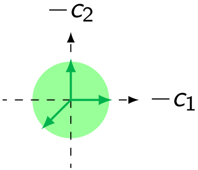

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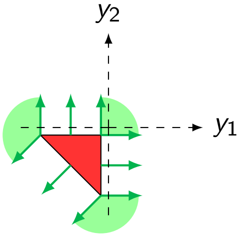

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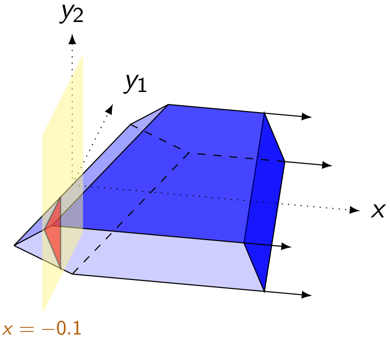

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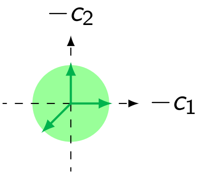

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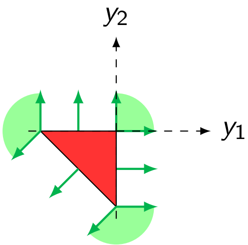

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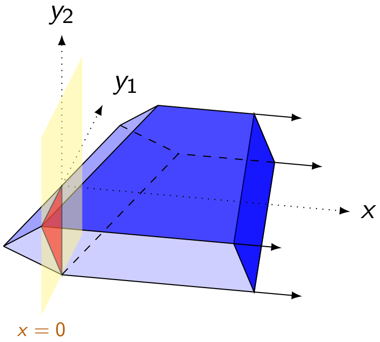

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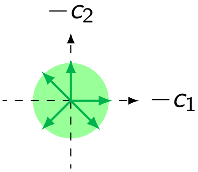

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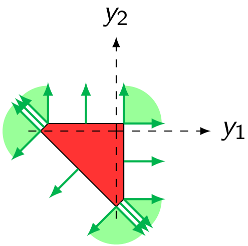

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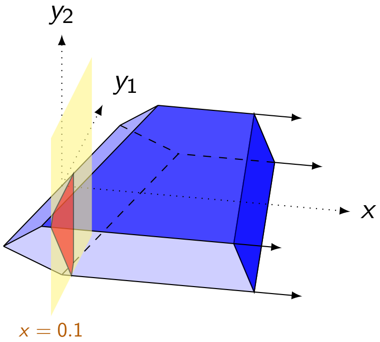

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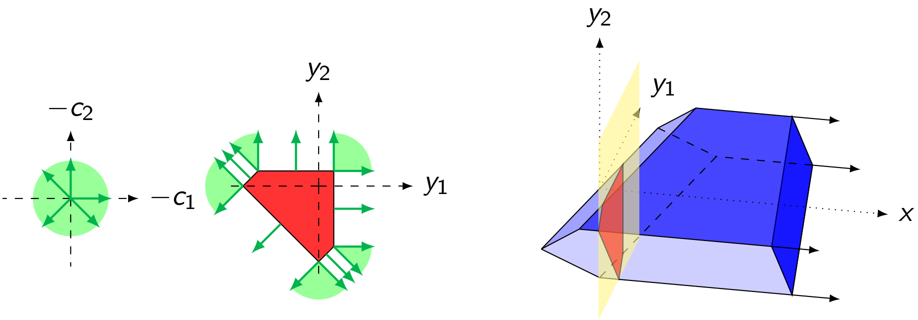

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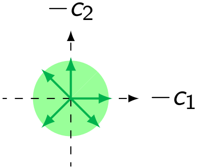

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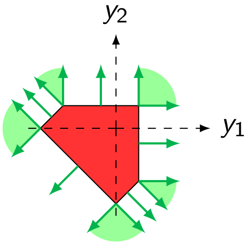

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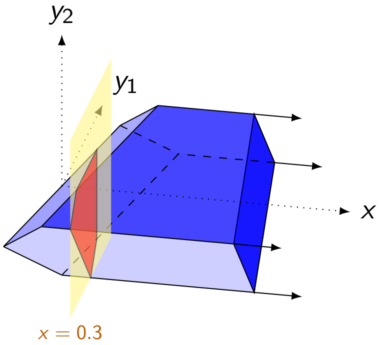

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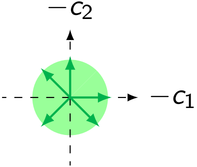
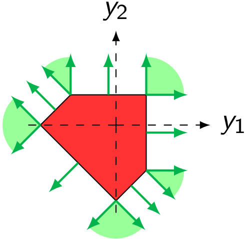
$$P := \{(x, y) \mid Ax + By \leqslant b\} \quad \text{and} \quad P_x := \{y \mid Ax + By \leqslant b\}$$

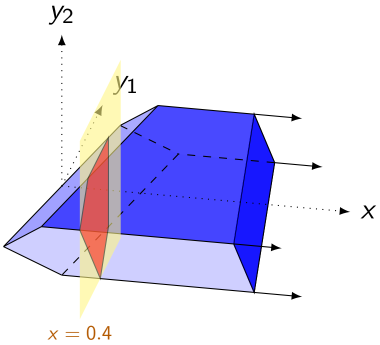
$$x = 0.4$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$



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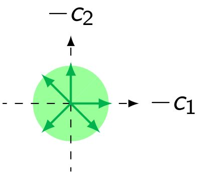
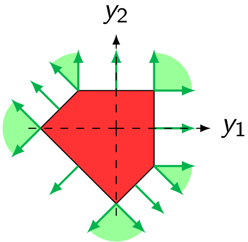
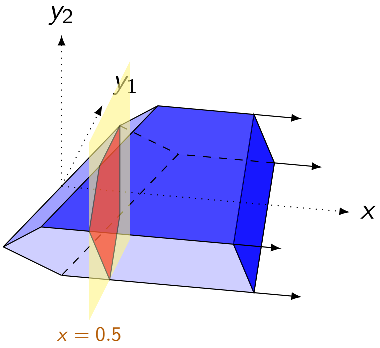
$x = 0.5$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$



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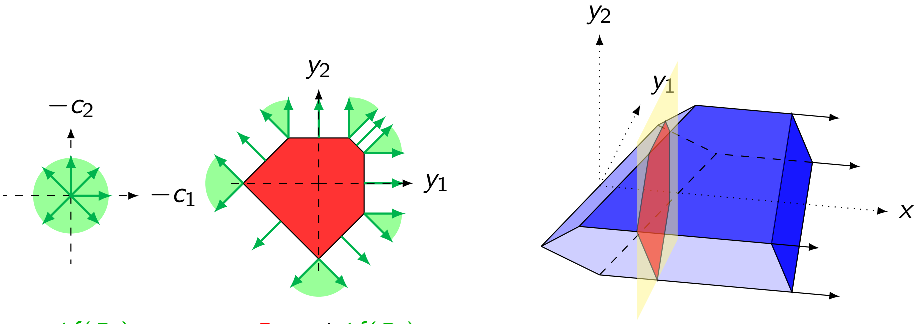

$\mathcal{N}(P_x)$ $\quad$ P_x and $\mathcal{N}(P_x)$ $\quad$ P and P_x

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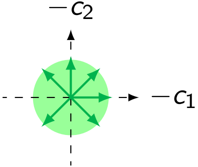
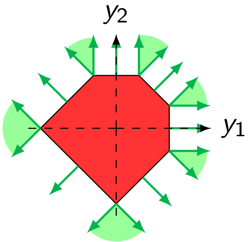
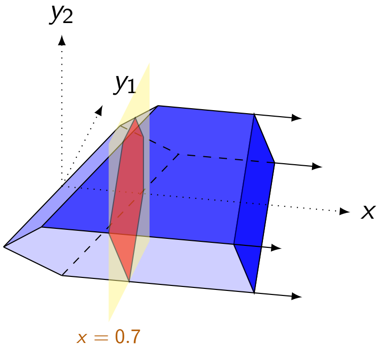


$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$



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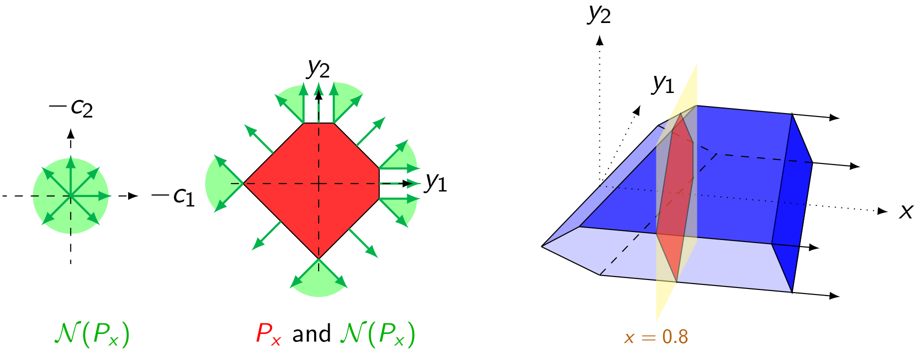
$x = 0.8$



$\mathcal{N}(P_x)$ P_x and $\mathcal{N}(P_x)$ P and P_x

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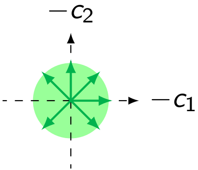
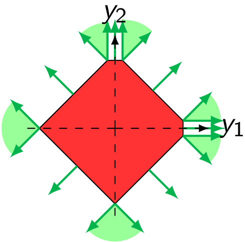
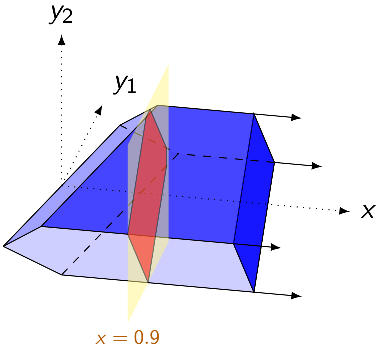
$x = 0.9$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$



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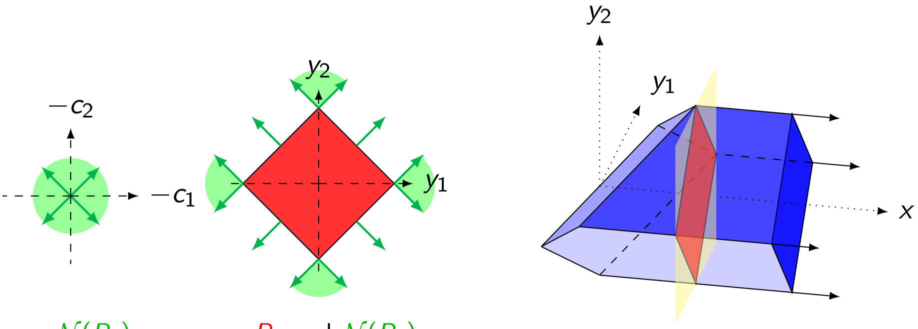
$x = 1$



$\mathcal{N}(P_x)$ — P_x and $\mathcal{N}(P_x)$ — P and P_x

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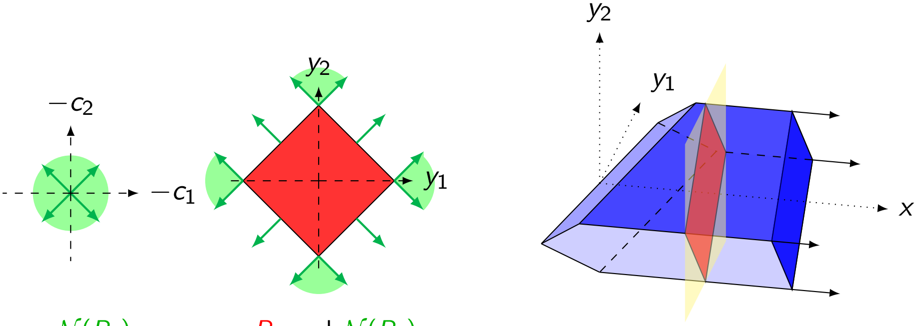
$$P := \{(x,y) \mid Ax + By \leqslant b\} \quad \text{and} \quad P_x := \{y \mid Ax + By \leqslant b\}$$

$$x = 1.1$$



$\mathcal{N}(P_x)$ $\quad$ P_x and $\mathcal{N}(P_x)$ $\quad$ P and P_x

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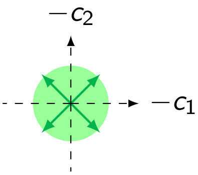
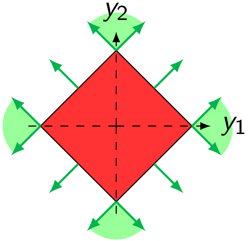
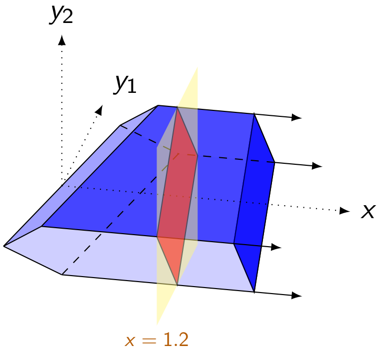
$x = 1.2$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$



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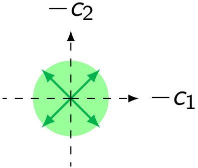
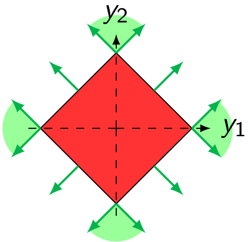
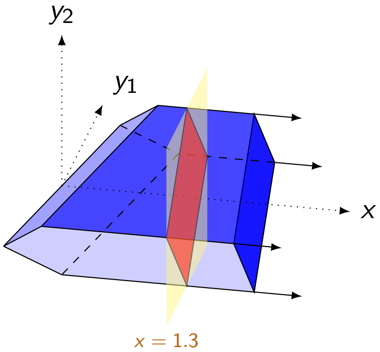
$x = 1.3$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$



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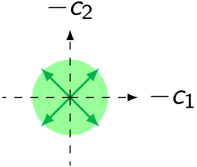
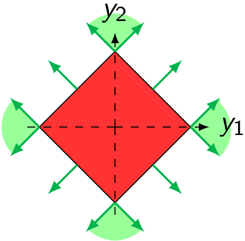
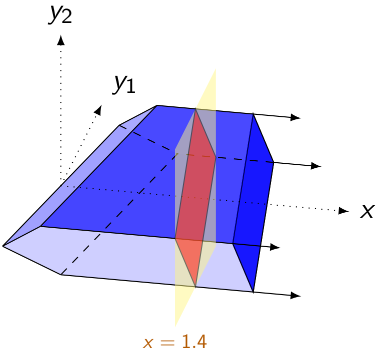


$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$



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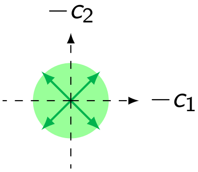
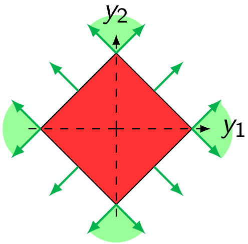
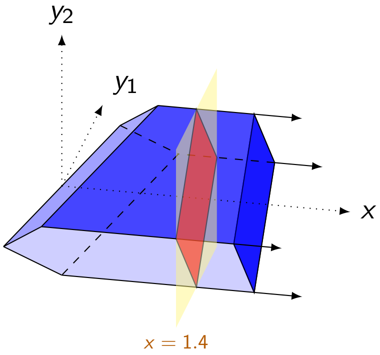


$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$



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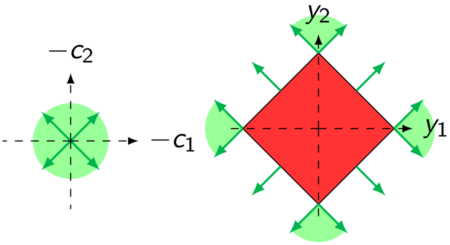
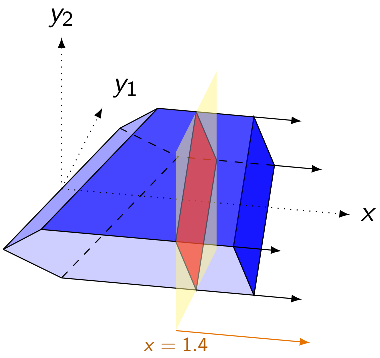


$\mathcal{N}(P_x)$ $\qquad$ P_x and $\mathcal{N}(P_x)$



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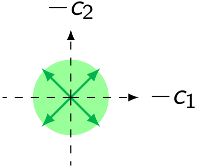
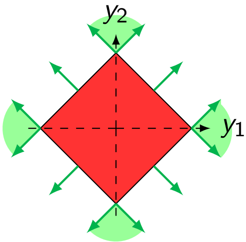
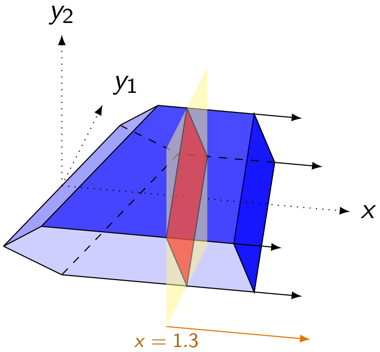


$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$



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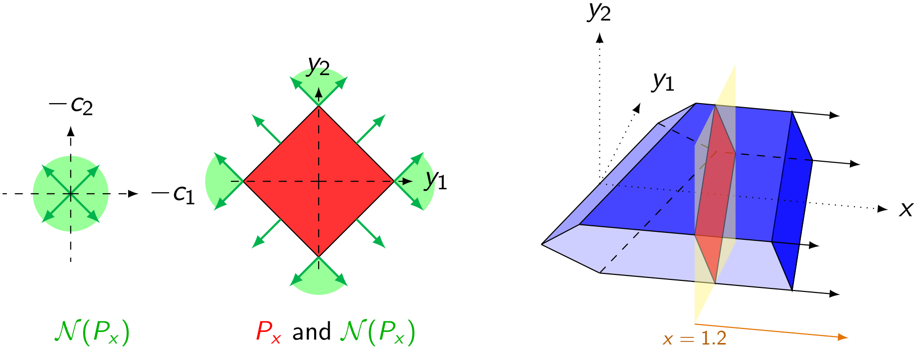

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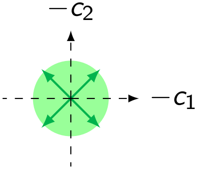
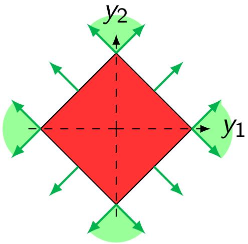
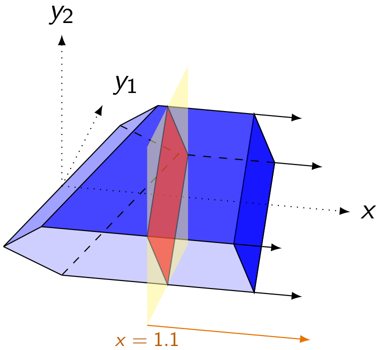


$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$



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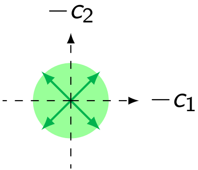
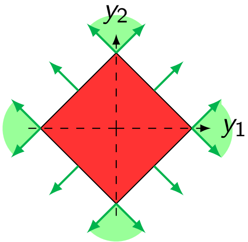
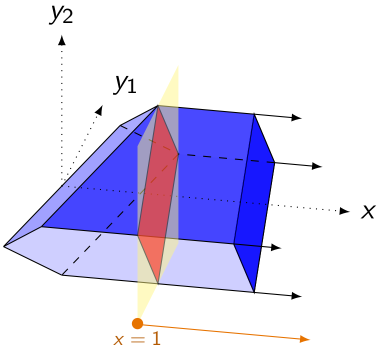
$x = 1$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$



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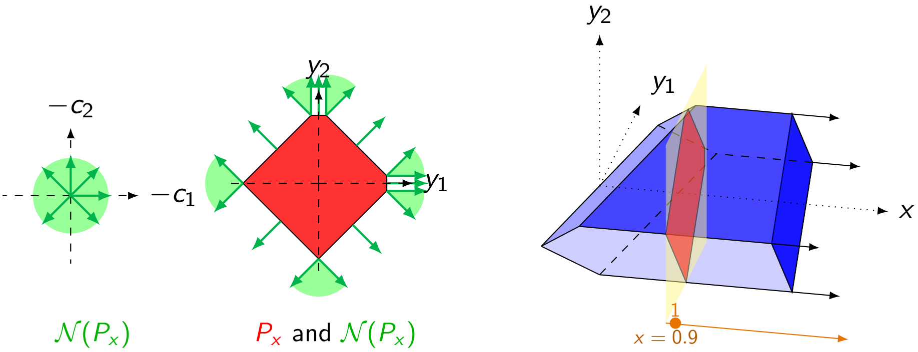

$\mathcal{N}(P_x)$ $\quad$ P_x and $\mathcal{N}(P_x)$ $\quad$ P and P_x

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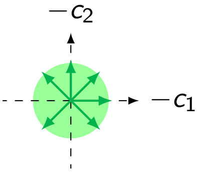
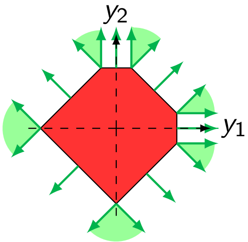
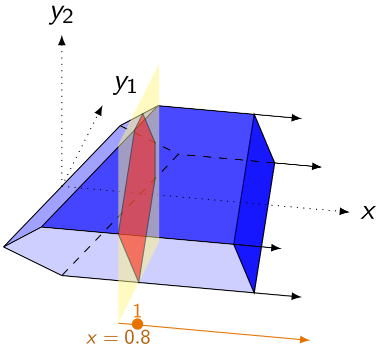


$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$



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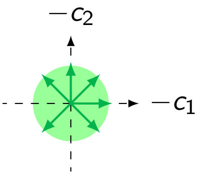
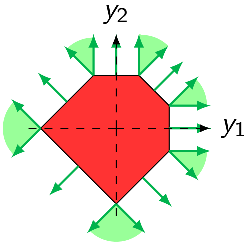
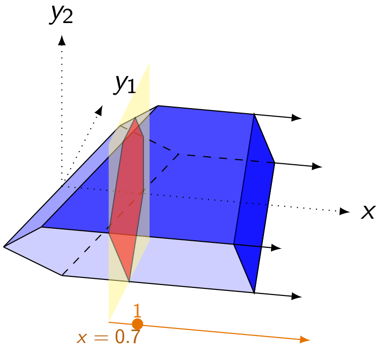


$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$



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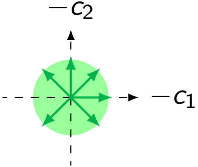
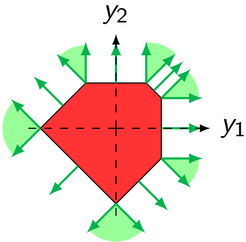
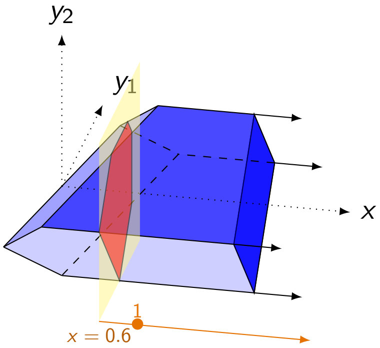


$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$



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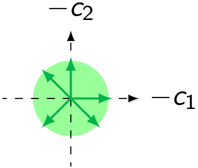
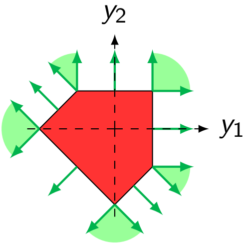


$\mathcal{N}(P_x)$



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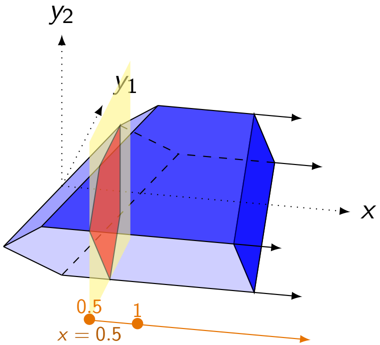

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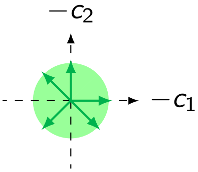
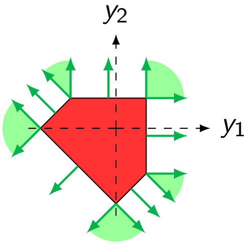
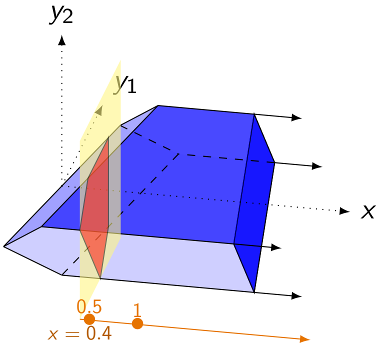


$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

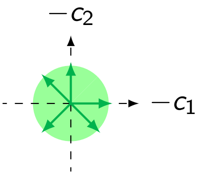


P and P_x

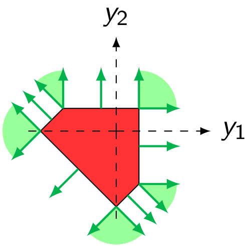
$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x, y) \mid Ax + By \leqslant b\} \quad \text{and} \quad P_x := \{y \mid Ax + By \leqslant b\}$$

$x = 0.3$



$\mathcal{N}(P_x)$



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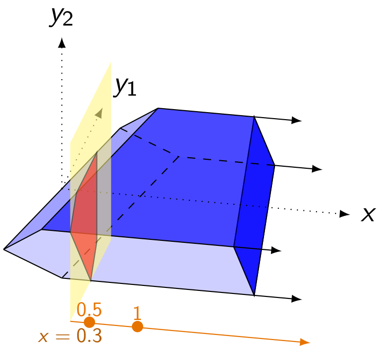
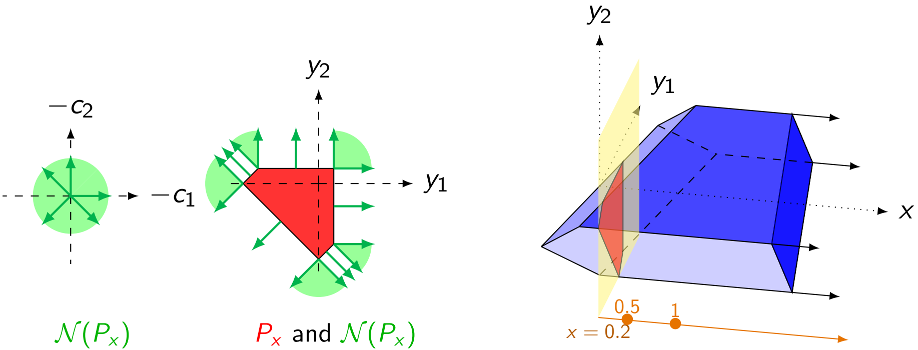


P and P_x

$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x, y) \mid Ax + By \leqslant b\} \quad \text{and} \quad P_x := \{y \mid Ax + By \leqslant b\}$$

$x = 0.2$



$\mathcal{N}(P_x)$ P_x and $\mathcal{N}(P_x)$ P and P_x

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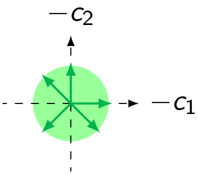
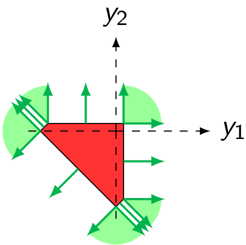
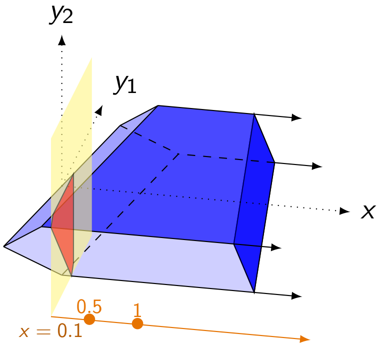
$x = 0.1$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$



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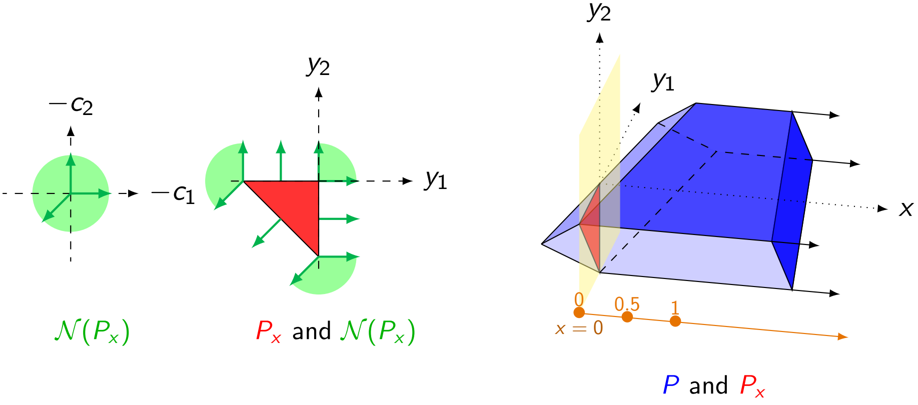
$x = 0$



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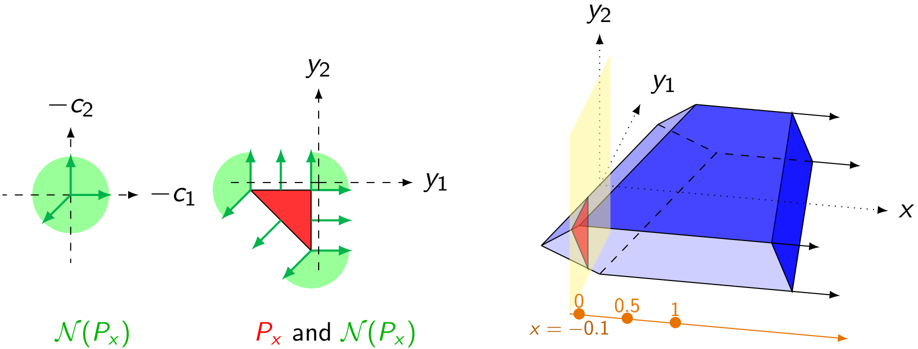
$$x = -0.1$$



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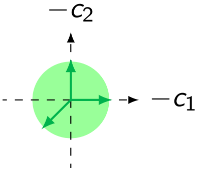
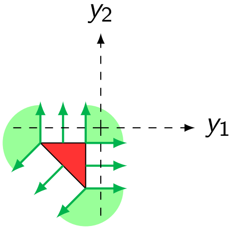


$\mathcal{N}(P_x)$



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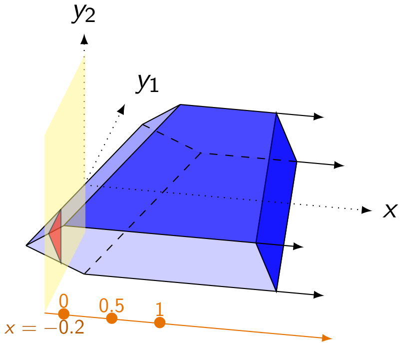

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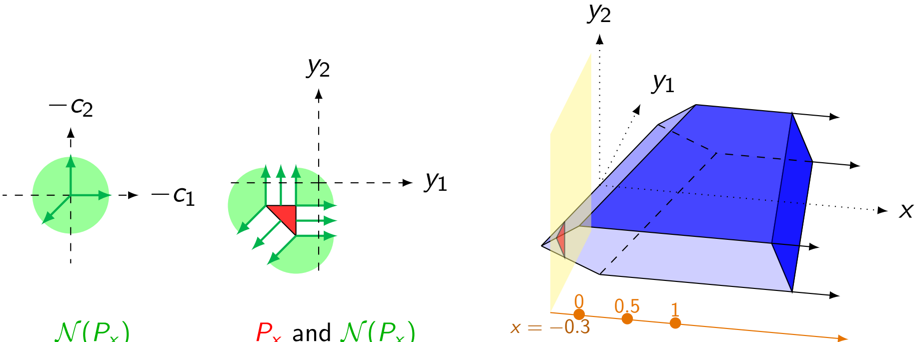

$\mathcal{N}(P_x)$ — P_x and $\mathcal{N}(P_x)$ — P and P_x

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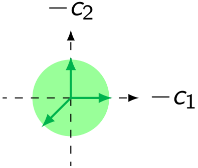
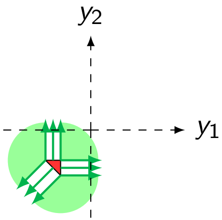


$\mathcal{N}(P_x)$



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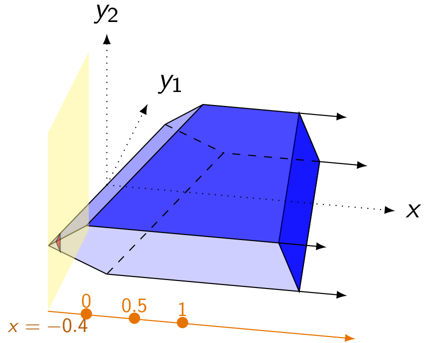


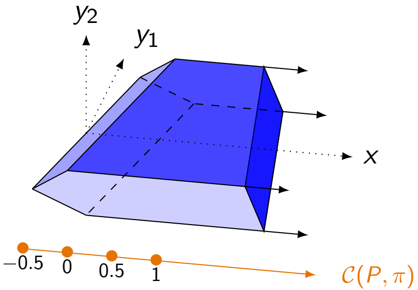
P and P_x

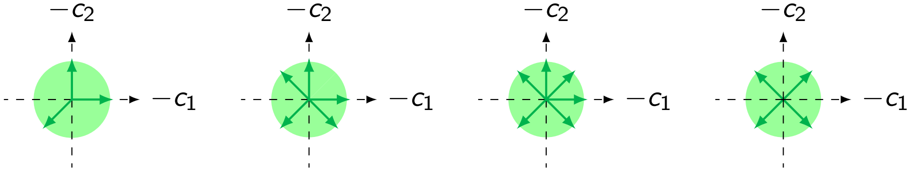
What are the constant regions of $\mathcal{N}(P_x)$?

Lemma

There exists a collection $\mathcal{C}(P,\pi)$ *called the* **chamber complex** *whose relative interior of cells are the constant regions of* $x \to \mathcal{N}(P_x)$.

For $\sigma \in \mathcal{C}(P,\pi)$ *and* $x, x' \in \mathrm{ri}(\sigma)$, $\mathcal{N}(P_x) = \mathcal{N}(P_{x'}) =: \mathcal{N}_\sigma$





$\mathcal{N}_\sigma$ for $\sigma = [-0.5, 0]$ $\mathcal{N}_\sigma$ for $\sigma = [0, 0.5]$ $\mathcal{N}_\sigma$ for $\sigma = [0.5, 1]$ $\mathcal{N}_\sigma$ for $\sigma = [1, +\infty)$

Chamber complex

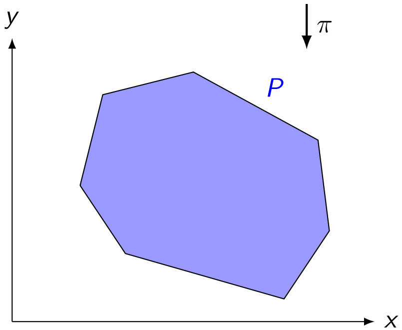
Definition (Billera, Sturmfels 92)

The *chamber complex* $\mathcal{C}(P, \pi)$ of P along π is

$$\mathcal{C}(P, \pi) := \{\sigma_{P,\pi}(x) \mid x \in \pi(P)\}$$

where

$$\sigma_{P,\pi}(x) := \bigcap_{F \in \mathcal{F}(P) \text{ s.t. } x \in \pi(F)} \pi(F)$$



where $\mathcal{F}(P)$ is the set of faces of P and π is the projection $(x, y) \to x$

$$\pi(E) := \{x \in \mathbb{R}^n \mid \exists y \in \mathbb{R}^m,\ (x, y) \in E\}$$

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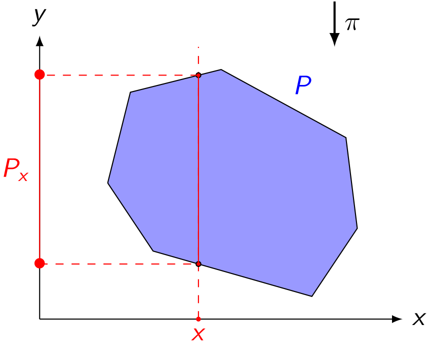

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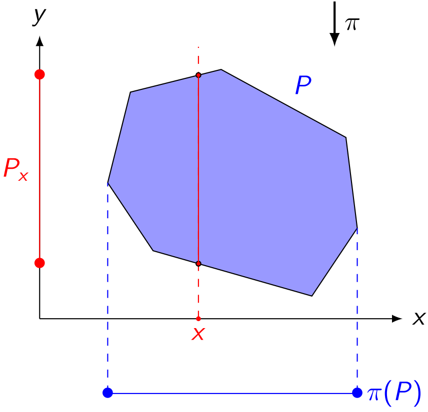

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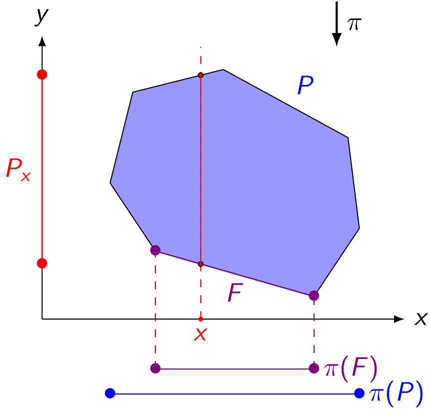

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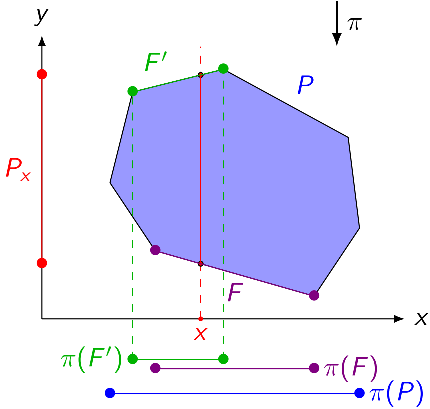

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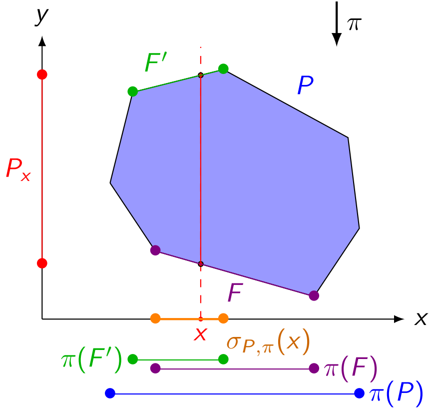

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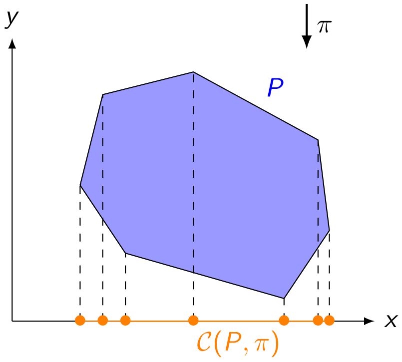

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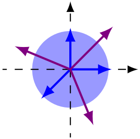
Common Refinement of Normal Fans

We can quantize **c** on each chamber.



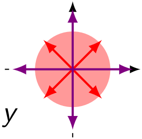
$\mathcal{N}_\sigma$ and $\check{\mathbf{c}}$

For all $x \in \mathrm{ri}(\sigma)$,

$$V(x) = \sum_{N \in \mathcal{N}_\sigma} p_N \min_{y \in P_x} \check{c}_N{}^\top y$$

For all $x \in \mathrm{ri}(\tau)$,

$$V(x) = \sum_{N \in \mathcal{N}_\tau} p_N \min_{y \in P_x} \check{c}_N{}^\top y$$



$\mathcal{N}_\tau$ and $\check{\mathbf{c}}$

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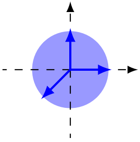

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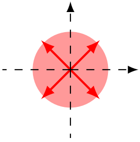

$\mathcal{N}_\tau$

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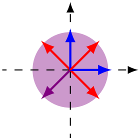
$$\mathcal{R} := \mathcal{N}_\sigma \wedge \mathcal{N}_\tau = \{N \cap N' \mid N \in \mathcal{N}_\sigma, N' \in \mathcal{N}_\tau\}$$



$\mathcal{N}_\sigma \wedge \mathcal{N}_\tau$

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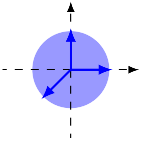

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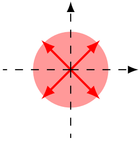

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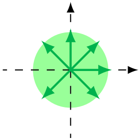

$\mathcal{R}$

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General cost $\mathbf{c}$ is equivalent to discrete cost $\check{\mathbf{c}}$ for all x

Theorem (Quantization of the cost distribution)

Let $\mathcal{R} = \bigwedge_{\sigma \in \mathcal{C}(P,\pi)} -\mathcal{N}_\sigma$*, then* **for all** $x \in \mathbb{R}^n$

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where $\check{p}_R := \mathbb{P}\big[\mathbf{c} \in \mathrm{ri}(R)\big]$ *and* $\check{c}_R := \mathbb{E}\big[\mathbf{c} \,|\, \mathbf{c} \in \mathrm{ri}(R)\big]$
Moreover, for all distributions of $\mathbf{c}$*,*
V *is affine on each cell of the chamber complex* $\mathcal{C}(P,\pi)$*.*

Bonus: This quantization method works for *every distribution of* $\mathbf{c}$!

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Bonus: This quantization method works for *every distribution of* $\mathbf{c}$!

Extension to multistage and stochastic constraints

Theorem

All results generalize to stochastic constraints with finite support and multistage

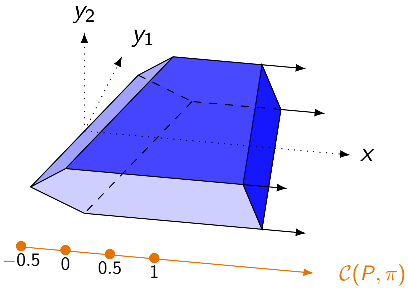
- $\rightsquigarrow$ *The regions where $(V_t)_t$ is affine do not depend on the $(\mathbf{c}_t)_t$*
- $\rightsquigarrow$ *We have an exact discretization method working for all $(\mathbf{c}_t)_t$*

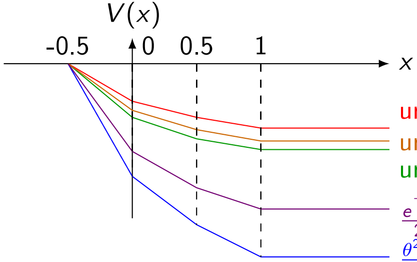
Idea of the proof :
Iterated chamber complexes

$$\mathcal{P}_{t,\xi} := \mathcal{C}(\mathbb{R}^{n_t} \times \mathcal{P}_{t+1} \wedge \mathcal{F}\big(P_t(\xi)\big), \pi_{x_{t-1}}^{x_{t-1},x_t})$$
$$\mathcal{P}_t := \bigwedge_{\xi_t \in \operatorname{supp} \xi_t} \mathcal{P}_{t,\xi}$$

Explicit computation of the example

$$V(x) = \mathbb{E}\begin{bmatrix} \min\limits_{y\in\mathbb{R}^2} & \mathbf{c}^\top y \\ \text{s.t.} & \|y\|_1 \leqslant 1 \\ & y_1 \leqslant x \\ & y_2 \leqslant x \end{bmatrix}$$





Different distributions of $\mathbf{c}$:

uniform on norm 1 ball

uniform on norm 2 ball

uniform on norm ∞ ball

$\frac{e^{-\frac{\|c\|_2^2}{2\gamma^2}}}{2\pi\gamma^2}dc$

$\frac{\theta^2 e^{-\theta\|c\|_1}}{4}dc$

Explicit formulas for usual distributions

Distribution	Uniform on polytope	Exponential	Gaussian
$d\mathbb{P}(c)$	$\frac{\mathbb{1}_{c\in Q}}{\mathrm{Vol}_d(Q)} d\mathcal{L}_{\mathrm{Aff}(Q)}(c)$	$\frac{e^{\theta^\top c}\mathbb{1}_{c\in K}}{\Phi_K(\theta)} d\mathcal{L}_{\mathrm{Aff}(K)}c$	$\frac{e^{-\frac{1}{2}c^\top M^{-2}c}}{(2\pi)^{\frac{m}{2}} \det M} dc$
Support	Polytope : Q	Cone : K	$\mathbb{R}^m$
$\mathbb{P}[\mathbf{c} \in S]$	$\frac{\mathrm{Vol}_d(S)}{\mathrm{Vol}_d(Q)}$	$\frac{\lvert\det(\mathrm{Ray}(S))\rvert}{\Phi_K(\theta)} \prod_{r\in\mathrm{Ray}(S)} \frac{1}{-r^\top\theta}$	$\mathrm{Ang}\left(M^{-1}S\right)$
$\mathbb{E}\left[\mathbf{c} \mid \mathbf{c} \in S\right]$	$\frac{1}{d}\sum_{v\in\mathrm{Vert}(S)} v$	$\left(\sum_{r\in\mathrm{Ray}(S)} \frac{-r_i}{r^\top\theta}\right)_{i\in[m]}$	$\frac{\sqrt{2}\Gamma(\frac{m+1}{2})}{\Gamma(\frac{m}{2})} M\,\mathrm{Ctr}\left(S \cap \mathbb{S}_{m-1}\right)$

These formulas are valid for S full dimensional simplex or simplicial cone.

Contents

Earlier and new complexity results

Volume of a polytope

$$\mathrm{Vol}\left(\{z \in \mathbb{R}^d \mid Az \leqslant b\}\right) \text{ or}$$
$$\mathrm{Vol}\left(\mathrm{Conv}(v_1, \cdots, v_n)\right)$$

- $\sharp P$-complete: Dyer and Frieze (1988)
- Polynomial for fixed dimension d: Barvinok (1994)

2-stage linear problem

$$\min_{x \in \mathbb{R}^n} c_0^\top x + \mathbb{I}_{Ax \leqslant b} + \mathbb{E}\Big[\min_{y \in \mathbb{R}^m} \mathbf{c}^\top y + \mathbb{I}_{\mathbf{T}x + \mathbf{W}y \leqslant \mathbf{h}}\Big]$$

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- Polynomial for fixed m ?

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- Polynomial for fixed m:
  FGL (2020)

Complexity result multistage

We can generalize to multistage by fixing several dimensions and the horizon.

Theorem (MSLP is polynomial for fixed dimensions)

Assume that $T \geqslant 3, n_2, \ldots, n_T$, $\sharp\big(\operatorname{supp}(\mathbf{A}_2, \mathbf{B}_2, \mathbf{b}_2)\big)$, $\cdots$,$\sharp\big(\operatorname{supp}(\mathbf{A}_T, \mathbf{B}_T, \mathbf{b}_T)\big)$ *are fixed integers*

and for all $t \in [T]$, $\mathbf{c}_t$ *conditionally to* $\{(\mathbf{A}_t, \mathbf{B}_t, \mathbf{b}_t) = (A, B, b)\}$ *is easily computable.*

Then, we can solve MSLP in polynomial time.

Conclusion

- MSLP with arbitrary cost distribution can be exactly discretized;
- new algebraic insights on the polyhedral structure of MSLP;
- analytical formulas for some usual distributions;
- fixed-parameter versions of 2SLP and MSLP are polynomial time.

Perspectives

$\rightsquigarrow$ New algorithms from the algebraic structure

$\rightsquigarrow$ Sensibility analysis to the distribution, link with nested distance;

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Thank you for listening ! Any question ?

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