Multistage stochastic optimization and polyhedral geometry

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Élucubrations scientifiques

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Modeling hydroelectric energy storage management

Modeling hydroelectric energy storage management

- u water hustled
- d demand
- c cost of unmet demand

$$\min_{u} \; c(d - u)$$
$$s.c. \;\; 0 \leqslant u \leqslant d$$

Modeling hydroelectric energy storage management

- u water hustled
- d demand
- c cost of unmet demand
- x_0/x_1 water in the reservoir
- $\overline{x}$ capacity of the reservoir

$$
\begin{aligned}
\min_{u} \quad & c(d - u) \\
s.c. \quad & 0 \leqslant u \leqslant d \\
& x_1 = x_0 - u \\
& 0 \leqslant x_0 \leqslant \overline{x},\ 0 \leqslant x_1 \leqslant \overline{x}
\end{aligned}
$$

Modeling hydroelectric energy storage management

- u water hustled
- d demand
- c cost of unmet demand
- x_0/x_1 water in the reservoir
- $\overline{x}$ capacity of the reservoir
- w rain and runoff

$$\begin{aligned} \min_{u} \quad & c(d-u) \\ s.c. \quad & 0 \leqslant u \leqslant d \\ & x_1 = x_0 - u + w \\ & 0 \leqslant x_0 \leqslant \overline{x},\ 0 \leqslant x_1 \leqslant \overline{x} \end{aligned}$$

Modeling hydroelectric energy storage management

- u water hustled
- d demand
- c cost of unmet demand
- x_0/x_1 water in the reservoir
- $\overline{x}$ capacity of the reservoir
- w rain and runoff
- v water evacuated by the valve

$$\begin{aligned} \min_{u,v}\ & c(d-u) \\ s.c.\ & 0 \leqslant u \leqslant d \\ & x_1 = x_0 - u + w - v \\ & 0 \leqslant x_0 \leqslant \overline{x},\ 0 \leqslant x_1 \leqslant \overline{x} \\ & 0 \leqslant v \end{aligned}$$

Modeling hydroelectric energy storage management

At step t

- u_t water hustled
- d_t demand
- c_t cost of unmet demand
- x_t water in the reservoir
- $\overline{x}$ capacity of the reservoir
- w_t rain and runoff
- v_t water evacuated by the valve

$$
\begin{aligned}
&\min_{u_t, v_t} \sum_{t=1}^{T} c_t(d_t - u_t) \\
s.c.\ &\forall t \in [T],\ 0 \leqslant u_t \leqslant d_t \\
&\forall t \in [T],\ x_{t+1} = x_t - u_t + w_t - v_t \\
&\forall t \in [T],\ 0 \leqslant x_t \leqslant \overline{x} \\
&\forall t \in [T],\ 0 \leqslant v_t
\end{aligned}
$$

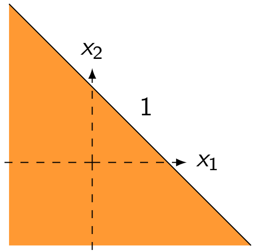
Linear Programming

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & c^\top x \\ \text{s.t.} \quad & Ax \leqslant b \end{aligned}$$

Example: $P = \{x \in \mathbb{R}^n \,|\, Ax \leqslant b\}$

$$A = \begin{pmatrix} 1 & 1 \\ & \\ & \\ & \\ & \\ & \\ & \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ \\ \\ \\ \\ \\ \\ \end{pmatrix}$$

$$\begin{aligned} x_1 + x_2 \leqslant 1 \qquad & (1) \\ & (2) \\ & (3) \\ & (4) \\ & (5) \\ & (6) \\ & (7) \end{aligned}$$



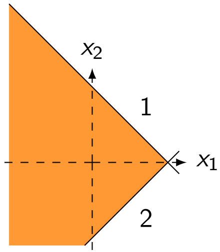
Linear Programming

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & c^\top x \\ \text{s.t.} \quad & Ax \leqslant b \end{aligned}$$

Example: $P = \{x \in \mathbb{R}^n \,|\, Ax \leqslant b\}$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ & \\ & \\ & \\ & \\ & \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 1 \\ \\ \\ \\ \\ \\ \end{pmatrix}$$

$$\begin{aligned} x_1 + x_2 &\leqslant 1 & (1) \\ x_1 - x_2 &\leqslant 1 & (2) \\ & & (3) \\ & & (4) \\ & & (5) \\ & & (6) \\ & & (7) \end{aligned}$$



Linear Programming

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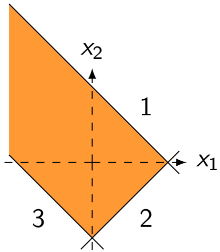
Example: $P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -1 & -1 \\ & \\ & \\ & \\ & \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \\ \\ \\ \\ \end{pmatrix}$$

$$\begin{aligned} x_1 + x_2 &\leqslant 1 && (1) \\ x_1 - x_2 &\leqslant 1 && (2) \\ -x_1 - x_2 &\leqslant 1 && (3) \\ & && (4) \\ & && (5) \\ & && (6) \\ & && (7) \end{aligned}$$



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Example: $P = \{x \in \mathbb{R}^n \,|\, Ax \leqslant b\}$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -1 & -1 \\ -1 & 1 \\ & \\ & \\ & \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ \\ \\ \\ \end{pmatrix}$$

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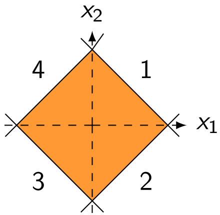


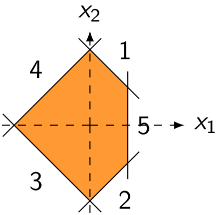
Linear Programming

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & c^\top x \\ \text{s.t.} \quad & Ax \leqslant b \end{aligned}$$

Example: $P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -1 & -1 \\ -1 & 1 \\ 1 & 0 \\ & \\ & \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0.5 \\ \\ \end{pmatrix}$$

$$\begin{aligned} x_1 + x_2 &\leqslant 1 && (1) \\ x_1 - x_2 &\leqslant 1 && (2) \\ -x_1 - x_2 &\leqslant 1 && (3) \\ -x_1 + x_2 &\leqslant 1 && (4) \\ x_1 &\leqslant 0.5 && (5) \\ & && (6) \\ & && (7) \end{aligned}$$



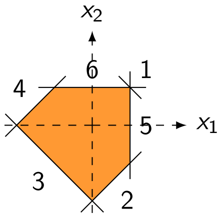
Linear Programming

$$\begin{array}{ll} \min\limits_{x \in \mathbb{R}^n} & c^\top x \\ \text{s.t.} & Ax \leqslant b \end{array}$$

Example: $P = \{x \in \mathbb{R}^n \,|\, Ax \leqslant b\}$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -1 & -1 \\ -1 & 1 \\ 1 & 0 \\ 0 & 1 \\ & \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0.5 \\ 0.5 \\ \end{pmatrix}$$

$$\begin{aligned} x_1 + x_2 &\leqslant 1 && (1) \\ x_1 - x_2 &\leqslant 1 && (2) \\ -x_1 - x_2 &\leqslant 1 && (3) \\ -x_1 + x_2 &\leqslant 1 && (4) \\ x_1 &\leqslant 0.5 && (5) \\ x_2 &\leqslant 0.5 && (6) \\ & && (7) \end{aligned}$$



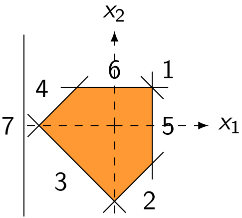
Linear Programming

$$\min_{x \in \mathbb{R}^n} \quad c^\top x$$
$$\text{s.t.} \quad Ax \leqslant b$$

Example: $P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -1 & -1 \\ -1 & 1 \\ 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0.5 \\ 0.5 \\ -1.2 \end{pmatrix}$$

$$\begin{aligned} x_1 + x_2 &\leqslant 1 && (1) \\ x_1 - x_2 &\leqslant 1 && (2) \\ -x_1 - x_2 &\leqslant 1 && (3) \\ -x_1 + x_2 &\leqslant 1 && (4) \\ x_1 &\leqslant 0.5 && (5) \\ x_2 &\leqslant 0.5 && (6) \\ x_1 &\geqslant -1.2 && (7) \end{aligned}$$



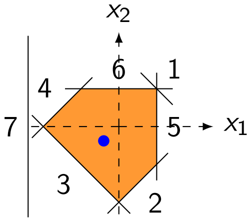
Active constraints

Definition

We denote by $\mathcal{I}(A, b)$, the collection of sets of active constraints as :

$$\mathcal{I}(A, b) = \{I_{A,b}(x) \mid Ax \leqslant b\}$$

with $I_{A,b}(x) := \{i \in [q] \mid A_i x = b_i\}$



$P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$

$I_{A,b}(x) = \emptyset$

To ease the notation, we write:

$$\mathcal{I}(A, b) = \{\emptyset, \qquad \}$$

Active constraints

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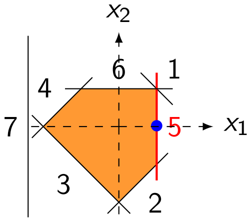


$P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$

$I_{A,b}(x) = \{5\}$

To ease the notation, we write:

$\mathcal{I}(A, b) = \{\emptyset, 5, \qquad \}$

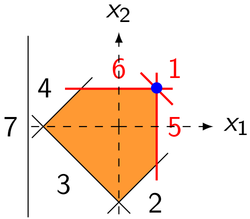
Active constraints

Definition

We denote by $\mathcal{I}(A,b)$, the collection of sets of active constraints as :

$$\mathcal{I}(A,b) = \{I_{A,b}(x) \mid Ax \leqslant b\}$$

with $I_{A,b}(x) := \{i \in [q] \mid A_i x = b_i\}$



$P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$

$I_{A,b}(x) = \{1, 5, 6\}$
To ease the notation, we write:

$\mathcal{I}(A,b) = \{\emptyset, 5, 156, \qquad\qquad\qquad\qquad\qquad\}$

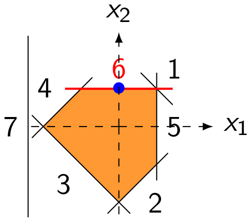
Active constraints

Definition

We denote by $\mathcal{I}(A, b)$, the collection of sets of active constraints as :

$$\mathcal{I}(A, b) = \{I_{A,b}(x) \mid Ax \leqslant b\}$$

with $I_{A,b}(x) := \{i \in [q] \mid A_i x = b_i\}$



$P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$

$I_{A,b}(x) = \{6\}$

To ease the notation, we write:

$\mathcal{I}(A, b) = \{\emptyset, 5, 156, 6, \qquad \}$

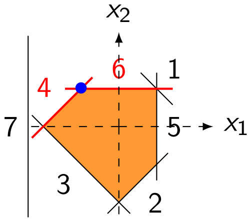
Active constraints

Definition

We denote by $\mathcal{I}(A,b)$, the collection of sets of active constraints as :

$$\mathcal{I}(A,b) = \{I_{A,b}(x) \mid Ax \leqslant b\}$$

with $I_{A,b}(x) := \{i \in [q] \mid A_i x = b_i\}$



$P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$

$I_{A,b}(x) = \{4, 6\}$

To ease the notation, we write:

$$\mathcal{I}(A,b) = \{\emptyset, 5, 156, 6, 46, \qquad \}$$

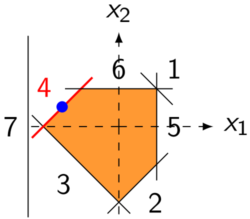
Active constraints

Definition

We denote by $\mathcal{I}(A, b)$, the collection of sets of active constraints as :

$$\mathcal{I}(A, b) = \{I_{A,b}(x) \mid Ax \leqslant b\}$$

with $I_{A,b}(x) := \{i \in [q] \mid A_i x = b_i\}$



$P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$

$I_{A,b}(x) = \{4\}$

To ease the notation, we write:

$$\mathcal{I}(A, b) = \{\emptyset, 5, 156, 6, 46, 4, \qquad\qquad \}$$

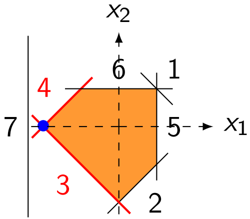
Active constraints

Definition

We denote by $\mathcal{I}(A,b)$, the collection of sets of active constraints as :

$$\mathcal{I}(A,b) = \{I_{A,b}(x) \mid Ax \leqslant b\}$$

with $I_{A,b}(x) := \{i \in [q] \mid A_i x = b_i\}$



$P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$

$I_{A,b}(x) = \{3,4\}$

To ease the notation, we write:

$$\mathcal{I}(A,b) = \{\emptyset, 5, 156, 6, 46, 4, 34, \qquad\qquad \}$$

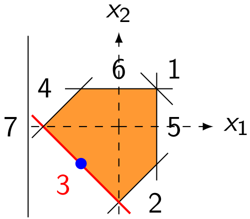
Active constraints

Definition

We denote by $\mathcal{I}(A,b)$, the collection of sets of active constraints as :

$$\mathcal{I}(A,b) = \{I_{A,b}(x) \mid Ax \leqslant b\}$$

with $I_{A,b}(x) := \{i \in [q] \mid A_i x = b_i\}$



$P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$

$I_{A,b}(x) = \{3\}$

To ease the notation, we write:

$$\mathcal{I}(A,b) = \{\emptyset, 5, 156, 6, 46, 4, 34, 3, \qquad \}$$

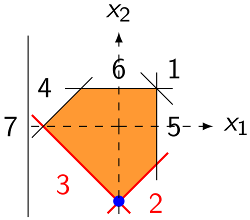
Active constraints

Definition

We denote by $\mathcal{I}(A,b)$, the collection of sets of active constraints as :

$$\mathcal{I}(A,b) = \{I_{A,b}(x) \mid Ax \leqslant b\}$$

with $I_{A,b}(x) := \{i \in [q] \mid A_i x = b_i\}$



$P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$

$I_{A,b}(x) = \{2,3\}$

To ease the notation, we write:

$$\mathcal{I}(A,b) = \left\{\emptyset, 5, 156, 6, 46, 4, 34, 3, 23, \qquad \right\}$$

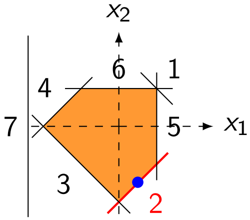
Active constraints

Definition

We denote by $\mathcal{I}(A, b)$, the collection of sets of active constraints as :

$$\mathcal{I}(A, b) = \{I_{A,b}(x) \mid Ax \leqslant b\}$$

with $I_{A,b}(x) := \{i \in [q] \mid A_i x = b_i\}$



$P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$

$I_{A,b}(x) = \{2\}$

To ease the notation, we write:

$$\mathcal{I}(A, b) = \{\emptyset, 5, 156, 6, 46, 4, 34, 3, 23, 2, \quad \}$$

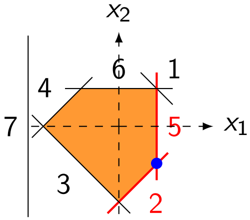
Active constraints

Definition

We denote by $\mathcal{I}(A, b)$, the collection of sets of active constraints as :

$$\mathcal{I}(A, b) = \{I_{A,b}(x) \mid Ax \leqslant b\}$$

with $I_{A,b}(x) := \{i \in [q] \mid A_i x = b_i\}$



$P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$

$I_{A,b}(x) = \{2, 5\}$

To ease the notation, we write:

$$\mathcal{I}(A, b) = \left\{\emptyset, 5, 156, 6, 46, 4, 34, 3, 23, 2, 25\right\}$$

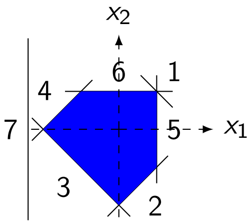
Faces

Definition

Let $I \in \mathcal{I}(A, b)$, we denote by P^I the face of P such that:

$$P^I = \{x \in P \mid A_I x = b_I\}$$

We have $\dim(P^I) = n - \mathrm{rg}(A_I)$
Example for $I = \emptyset$



P, P^I

Faces

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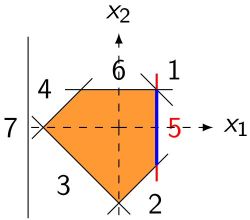
Example for $I = \{5\}$



P, P^I

Faces

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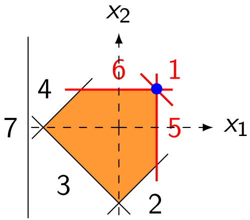
Example for $I = \{1, 5, 6\}$



P, P^I

Faces

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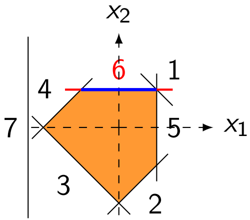
Let $I \in \mathcal{I}(A, b)$, we denote by P^I the face of P such that:

$$P^I = \{x \in P \mid A_I x = b_I\}$$

We have $\dim(P^I) = n - \mathrm{rg}(A_I)$
Example for $I = \{6\}$



P, P^I

Faces

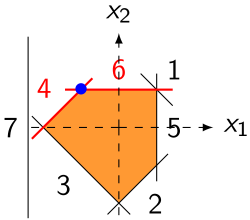
Definition

Let $I \in \mathcal{I}(A, b)$, we denote by P^I the face of P such that:

$$P^I = \{x \in P \mid A_I x = b_I\}$$

We have $\dim(P^I) = n - \mathrm{rg}(A_I)$

Example for $I = \{4, 6\}$



P, P^I

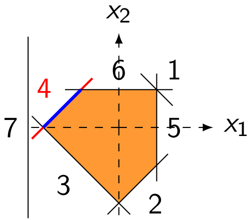
Faces

Definition

Let $I \in \mathcal{I}(A, b)$, we denote by P^I the face of P such that:

$$P^I = \{x \in P \mid A_I x = b_I\}$$

We have $\dim(P^I) = n - \mathrm{rg}(A_I)$
Example for $I = \{4\}$



P, P^I

Faces

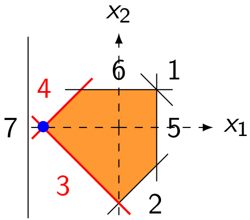
Definition

Let $I \in \mathcal{I}(A, b)$, we denote by P^I the face of P such that:

$$P^I = \{x \in P \mid A_I x = b_I\}$$

We have $\dim(P^I) = n - \mathrm{rg}(A_I)$

Example for $I = \{3, 4\}$



P, P^I

Faces

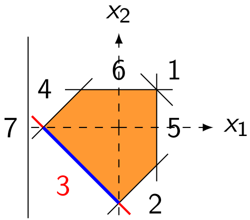
Definition

Let $I \in \mathcal{I}(A, b)$, we denote by P^I the face of P such that:

$$P^I = \{x \in P \mid A_I x = b_I\}$$

We have $\dim(P^I) = n - \mathrm{rg}(A_I)$

Example for $I = \{3\}$



P, P^I

Faces

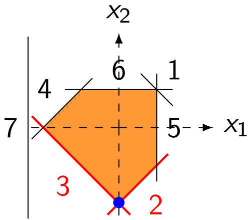
Definition

Let $I \in \mathcal{I}(A, b)$, we denote by P^I the face of P such that:

$$P^I = \{x \in P \mid A_I x = b_I\}$$

We have $\dim(P^I) = n - \mathrm{rg}(A_I)$

Example for $I = \{2, 3\}$



P, P^I

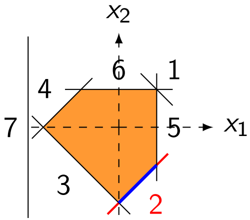
Faces

Definition

Let $I \in \mathcal{I}(A, b)$, we denote by P^I the face of P such that:

$$P^I = \{x \in P \mid A_I x = b_I\}$$

We have $\dim(P^I) = n - \mathrm{rg}(A_I)$
Example for $I = \{2\}$



P, P^I

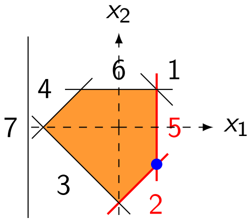
Faces

Definition

Let $I \in \mathcal{I}(A, b)$, we denote by P^I the face of P such that:

$$P^I = \{x \in P \mid A_I x = b_I\}$$

We have $\dim(P^I) = n - \mathrm{rg}(A_I)$
Example for $I = \{2, 5\}$



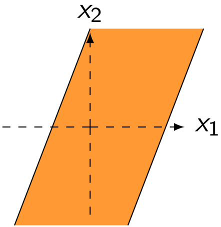
P, P^I

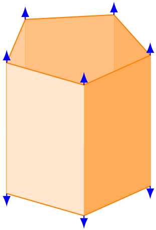
Polyhedra without any vertex ?

Definition (Lineality space)

$\mathrm{Lin}(C) := \{u \in C \mid \forall t \in \mathbb{R},\ \forall x \in c,\ x + tu \in C\}.$

$$\mathrm{Lin}\left(\{x \mid Ax \leqslant b\}\right) = \mathrm{Ker}(A)$$





Bases and Vertices

Let $P = \{x \in \mathbb{R}^n | Ax \leqslant b\}$ with $A \in \mathbb{R}^{p \times n}$ and $b \in \mathbb{R}^p$.

Definition

A basis B is a subset of $[p]$ such that $A_B = (A_{i,j})_{i \in B, 1 \leqslant j \leqslant n}$ is invertible.
A vertex of P is a face of dimension 0. $\mathrm{Vert}(P)$ is the set of vertices.

$$\begin{aligned} \mathrm{Vert}(P) \neq \emptyset &\iff A \text{ admits at least one basis} \\ &\iff \mathrm{rg}(A) = n \\ &\iff \mathrm{Lin}(P) = \{0\} \end{aligned}$$

Under this assumption,
For every $I \in \mathcal{I}(A, b)^{\max}$, we can extract a basis B_I and $P^I = \{A_{B_I}^{-1} b_{B_I}\}$.
If $c \notin \mathrm{Lin}(P)^\perp = \mathrm{Im}(A^\top)$, $\min_{x \in P} c^\top x = -\infty$.
Otherwise, we can write $P = P_0 + \mathrm{Lin}(P)$ with $\mathrm{Lin}(P_0) = \{0\}$:
We make this assumption without loss of generality

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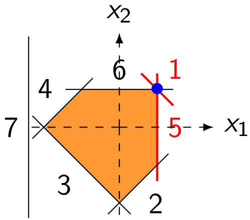
Simplex method

Geometrically:
follow a path on the polyhedron from vertex to vertex



Combinatorially:
pivoting from basis to basis

$$B_1 = \{1, 5\}$$

Simplex method

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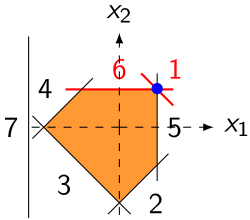

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$$B_2 = \{1, 6\}$$

Simplex method

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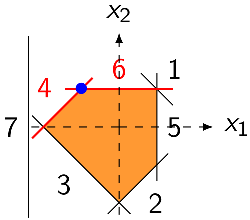

Combinatorially:
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$$B_1 = \{1, 5\}$$
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$$B_3 = \{4, 6\}$$

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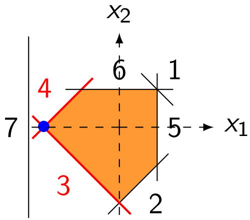

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Modeling hydroelectric energy storage management

At step t

- u_t water hustled
- d_t demand
- c_t cost of unmet demand
- x_t water in the reservoir
- $\overline{x}$ capacity of the reservoir
- w_t rain and runoff
- v_t water evacuated by the valve

$$
\begin{aligned}
&\min_{u_t, v_t} \quad \sum_{t=1}^{T} c_t(d_t - u_t) \\
s.c.\ &\forall t \in [T],\ 0 \leqslant u_t \leqslant d_t \\
&\forall t \in [T],\ x_{t+1} = x_t - u_t + w_t - v_t \\
&\forall t \in [T],\ 0 \leqslant x_t \leqslant \overline{x} \\
&\forall t \in [T],\ 0 \leqslant v_t
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Modeling hydroelectric energy storage management

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$$\min_{\boldsymbol{u}_t, \boldsymbol{v}_t} \mathbb{E}\Big[\sum_{t=1}^{T} \boldsymbol{c}_t(\boldsymbol{d}_t - \boldsymbol{u}_t)\Big]$$

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&\forall t \in [T],\ \sigma(\boldsymbol{u}_t, \boldsymbol{v}_t) \subset \sigma(\boldsymbol{d}_\tau, \boldsymbol{w}_\tau)_{\tau \leqslant t}
\end{aligned}
$$

Multistage stochastic linear programming (MSLP)

$$
\begin{aligned}
\min_{(\boldsymbol{x}_t)_{t\in[T]}} \quad & \mathbb{E}\Big[\sum_{t=1}^{T} \boldsymbol{c}_t^\top \boldsymbol{x}_t\Big] & \\
\text{s.t.} \quad & \boldsymbol{A}_t\boldsymbol{x}_t + \boldsymbol{B}_t\boldsymbol{x}_{t-1} \leqslant \boldsymbol{b}_t & \forall t \in [T] \\
& \sigma(\boldsymbol{x}_t) \subset \sigma(\boldsymbol{c}_\tau, \boldsymbol{A}_\tau, \boldsymbol{B}_\tau, \boldsymbol{b}_\tau)_{\tau\leqslant t} & \forall t \in [T] \\
& \boldsymbol{x}_0 \equiv x_0 \text{ given} &
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$\boldsymbol{\xi}_t = (\boldsymbol{c}_t, \boldsymbol{A}_t, \boldsymbol{B}_t, \boldsymbol{b}_t)_{t\in[T]}$ is assumed to be stagewise independent.

We set $V_{T+1} \equiv 0$ and:

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V_t(x_{t-1}) := \mathbb{E}\big[\hat{V}_t(x_{t-1}, \boldsymbol{\xi}_t)\big] := \mathbb{E}\begin{bmatrix} \min_{x_t\in\mathbb{R}^{n_t}} & \boldsymbol{c}_t^\top x_t + V_{t+1}(x_t) \\ \text{s.t.} & \boldsymbol{A}_t x_t + \boldsymbol{B}_t x_{t-1} \leqslant \boldsymbol{b}_t \end{bmatrix}
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➥ How to deal with continuous distributions ?

Multistage stochastic linear programming (MSLP)

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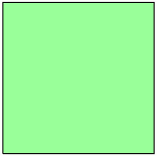
Quantization of a MSLP

Real problem

$$V_t(x) = \mathbb{E}\left[\hat{V}_t\left(x, \xi_t\right)\right] = \mathbb{E}\left[\begin{array}{ll} \min\limits_{y \in \mathbb{R}^{n_t}} & \boldsymbol{c}_t^\top y + V_{t+1}(y) \\ \text{s.t.} & \boldsymbol{A}_t y + \boldsymbol{B}_t x \leqslant \boldsymbol{b}_t \end{array}\right]$$



ξ_t continuous

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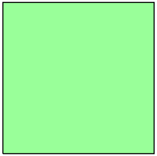
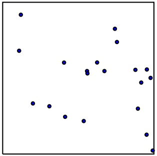


ξ_t continuous

Sample Average Approximation (SAA)

$$V_{t,N}^{SAA}(x) := \frac{1}{N}\sum_{k=1}^{N} \hat{V}_t\left(x, \xi^k\right)$$

$\xi^1, \cdots, \xi^N$ drawn by Monte Carlo



SAA $N = 20$

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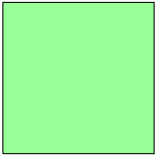

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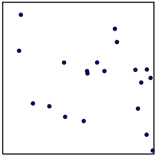
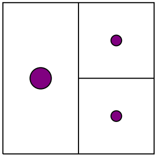


SAA $N = 20$

Partition-based

$$V_{t,\mathcal{P}}(x) := \sum_{P \in \mathcal{P}} \check{p}_{t,P}\, \hat{V}_t(x, \check{\xi}_{t,P})$$

with $\check{p}_{t,P} := \mathbb{P}\left[\boldsymbol{\xi}_t \in P\right]$ and $\check{\xi}_{t,P} := \mathbb{E}\left[\boldsymbol{\xi}_t \,|\, \boldsymbol{\xi}_t \in P\right]$



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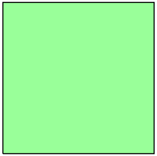

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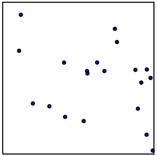

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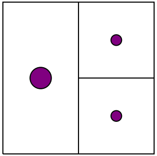
If $\xi \mapsto \hat{V}(x, \xi)$ is convex, $V_{t,\mathcal{P}}(x) \leqslant V_t(x)$.



Partition-based

Exact quantization

Definition

A MSLP admits a local exact quantization at time t on x if there exists a finitely supported $(\check{\xi}_t)_{t\in[T]}$ *i.e.* such that

$$V_t(x) = \mathbb{E}\big[\hat{V}_t(x, \xi_t)\big] = \mathbb{E}\big[\hat{V}_t(x, \check{\xi}_t)\big].$$

We call an exact quantization

- uniform if it is locally exact at all $x \in \mathbb{R}^{n_t}$, and all $t \in [T]$.
- universal if there exists a partition $\mathcal{P}_{t,x}$ such that the induced quantization is exact at time t on x, for all distributions of $(\xi_\tau)_{\tau\in[T]}$.

Questions:

1. Under which condition does there exist an exact quantization ?
2. Can we construct a uniform and universal exact quantization ?

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A first counter example

Assume $V_{t+1} \equiv 0$ and denote $V := V_t$, $\hat{V} := \hat{V}_t$ and $\boldsymbol{\xi} := \boldsymbol{\xi}_t$ for now.

Let $\boldsymbol{A} = (-\boldsymbol{u})$, $\boldsymbol{B} \equiv (0)$, $\boldsymbol{b} \equiv (-1)$ where $\boldsymbol{u} \sim \mathcal{U}([1,2])$.

$$\hat{V}(x,\xi) = \begin{array}{ll} \min\limits_{y \in \mathbb{R}} & y \\ \text{s.t.} & uy \geqslant 1 \end{array} = \frac{1}{u}$$

By strict convexity, for all partition $\mathcal{P}$

$$\sum_{P \in \mathcal{P}} \check{p}_P \hat{V}(x, \check{\xi}_P) < V(x) = \mathbb{E}\left[\frac{1}{\boldsymbol{u}}\right]$$

with $\check{p}_P = \mathbb{P}[\boldsymbol{\xi} \in P]$, $\check{\xi}_P = \mathbb{E}[\boldsymbol{\xi} \mid \boldsymbol{\xi} \in P]$.

➥ There is no partition-based local, neither uniform nor universal, exact quantization result for $\boldsymbol{A}$ non-finitely supported.

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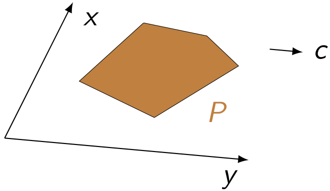
Uniform exact quantization and polyhedrality

$$\hat{V}(x,\xi) = \min_{y\in\mathbb{R}^m} \ c^\top y$$
$$\text{s.t.} \quad Ay + Bx \leqslant h$$



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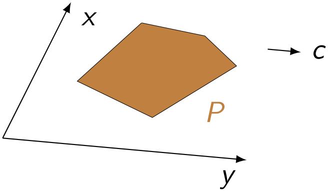

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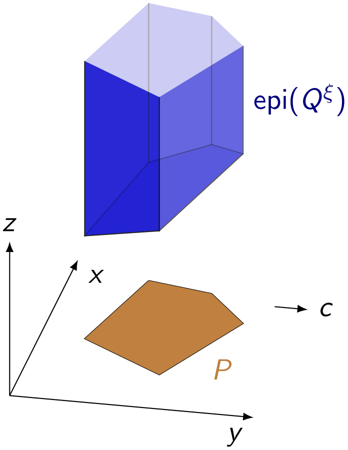

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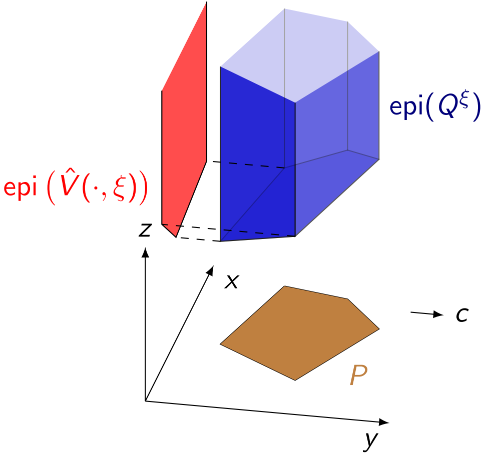

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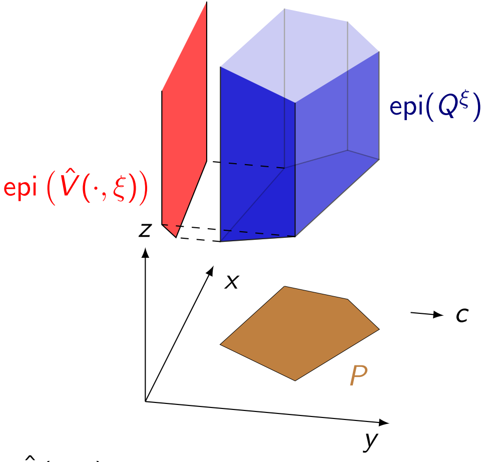

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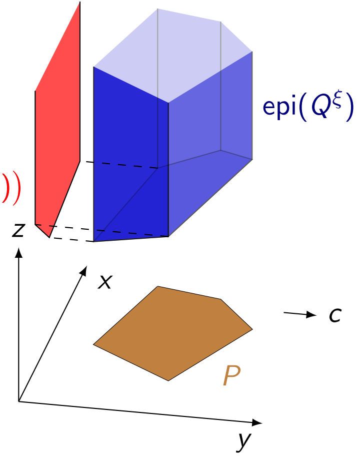

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- ➥ If the noise is finitely supported, then V is polyhedral
- ➥ Existence of uniform exact quantization implies polyhedrality of V.

Counter examples with stochastic constraints

Stochastic $\boldsymbol{B}$

$$V(x) = \mathbb{E}\begin{bmatrix} \min\limits_{y\in\mathbb{R}^m} & y \\ \text{s.t.} & \boldsymbol{u}x - y \leqslant 0 \\ & y \geqslant 1 \end{bmatrix}$$

$$= \mathbb{E}\Big[\max(\boldsymbol{u}x, 1)\Big]$$

$$= \begin{cases} 1 & \text{if } x \leqslant 1 \\ \frac{x}{2} + \frac{1}{2x} & \text{if } x \geqslant 1 \end{cases}$$

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	$\boldsymbol{A}$	$(\boldsymbol{B}, \boldsymbol{b})$	$\boldsymbol{c}$
Local	×	?	?
Uniform	×	×	?

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Uniform	$\times$	$\times$	?

Theorem (GAPM, FL 2022)

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Theorem (Exact quantization, FGL 2021)

If A, B and b are deterministic,
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Contents

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Illustrative running example:

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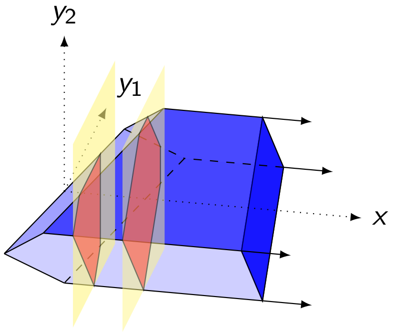

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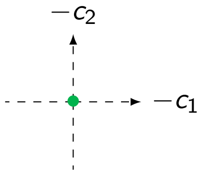
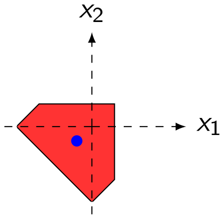


$N_{P_x}(y)$ for $x = 0.3$



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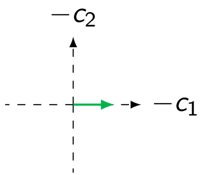

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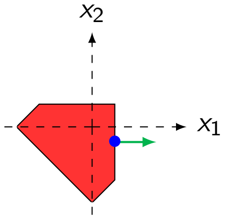

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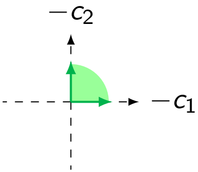

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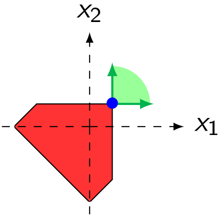

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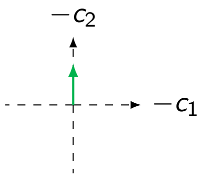

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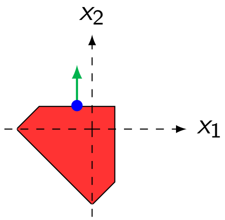

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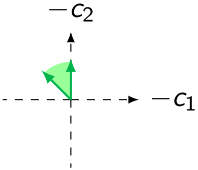

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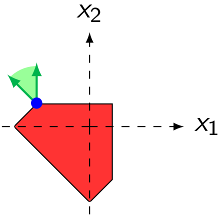

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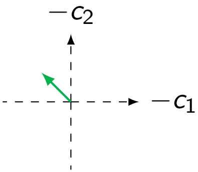

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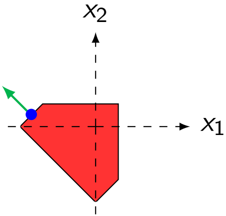

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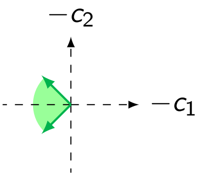

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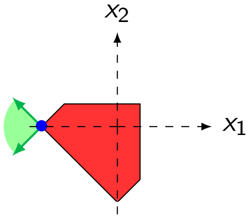

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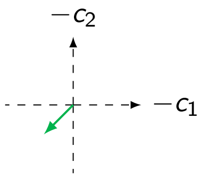

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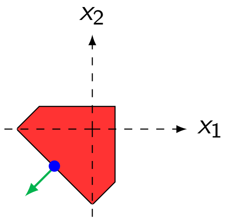

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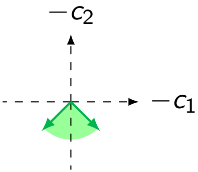

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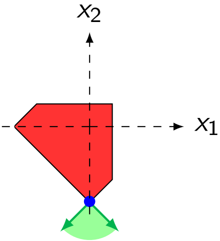

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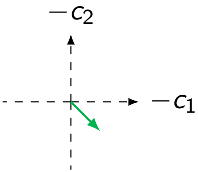

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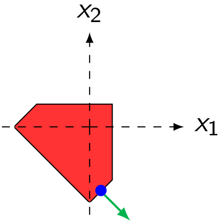

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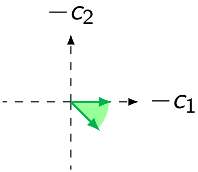

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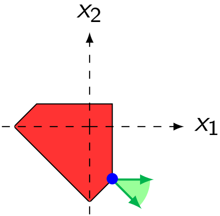

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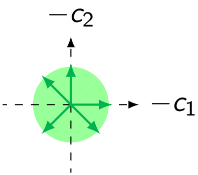
Proposition

If P_x *is bounded,* $\{\mathrm{ri}(N) \mid N \in \mathcal{N}(P_x)\}$ *is a partition of* $\mathbb{R}^m$.



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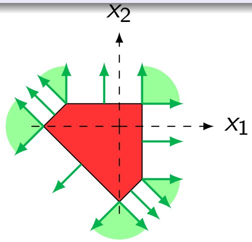
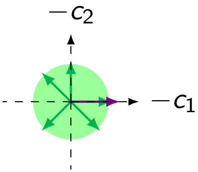


P_x and $\mathcal{N}(P_x)$ for $x = 0.3$

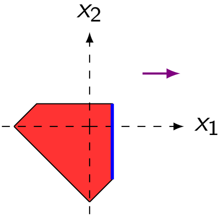
$\mathcal{N}(P_x)$: partition of cost coherent with the min

$$V(x) = \mathbb{E}\big[\min_{y \in P_x} \boldsymbol{c}^\top y\big]$$

For any $N \in \mathcal{N}(P_x)$, $-c \mapsto \arg\min_{y \in P_x} c^\top y$ is constant for all $-c \in \mathrm{ri}(N)$.



Cost $-c$ and $\mathcal{N}(P_x)$ for $x = 0.3$



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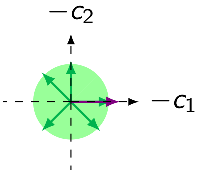

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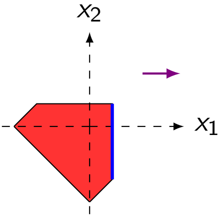

P_x for $x = 0.3$

$\mathcal{N}(P_x)$: partition of cost coherent with the min

$$V(x) = \mathbb{E}\big[\min_{y \in P_x} \boldsymbol{c}^\top y\big]$$

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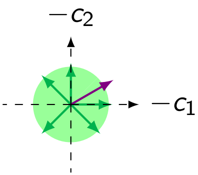

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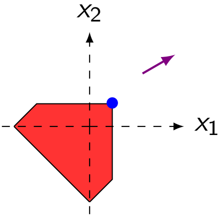

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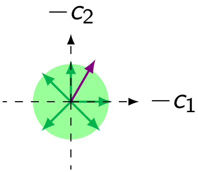

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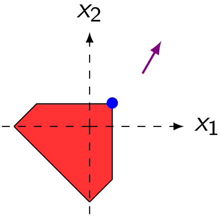

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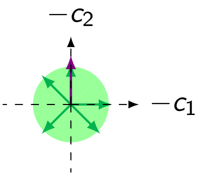

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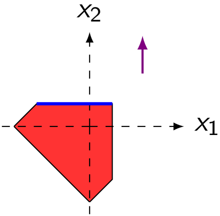

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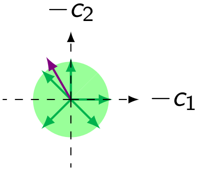

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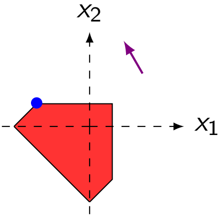

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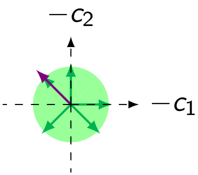

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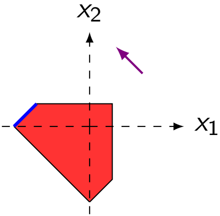

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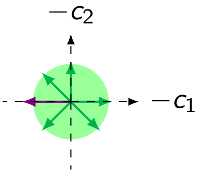

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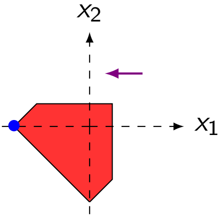

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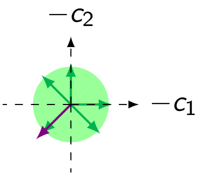

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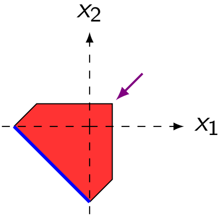

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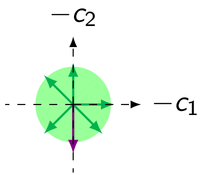

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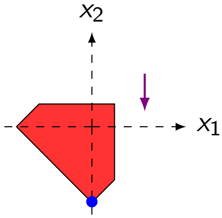

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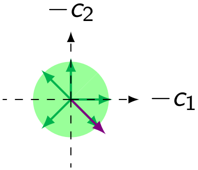

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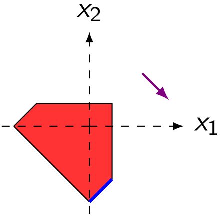

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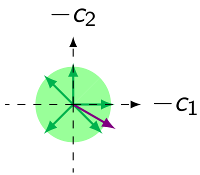

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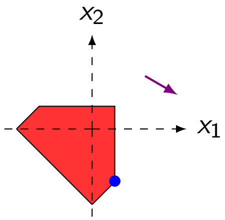

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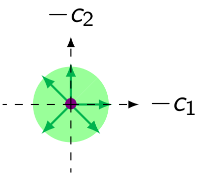

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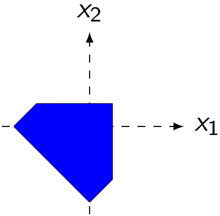

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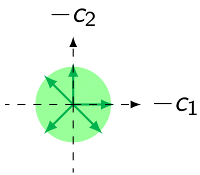

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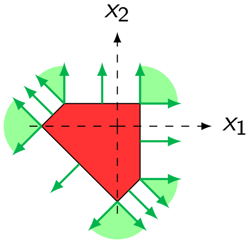

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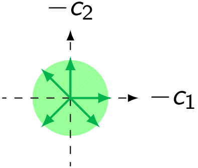
Local and universal exact quantization for $\boldsymbol{c}$

$$
\begin{aligned}
V(x) &= \mathbb{E}\Big[\min_{y \in P_x} \boldsymbol{c}^\top y\Big] \\
&= \sum_{N \in \mathcal{N}(P_x)} \mathbb{E}\Big[\mathbb{1}_{\boldsymbol{c} \in -\operatorname{ri} N} \min_{y \in P_x} \boldsymbol{c}^\top y\Big]
\end{aligned}
$$



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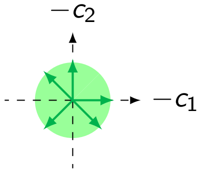
$$= \sum_{N \in \mathcal{N}(P_x)} \mathbb{E}\big[\mathbb{1}_{\boldsymbol{c} \in -\operatorname{ri} N} \boldsymbol{c}^\top\big] y_N(x)$$



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\end{aligned}
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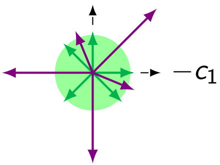

$\mathcal{N}(P_x)$ and $p_N \check{c}_N$ for $x = 0.3$

For $N \in \mathcal{N}(P_x)$,

$$
\begin{aligned}
p_N &:= \mathbb{P}\big[\boldsymbol{c} \in -\operatorname{ri} N\big] \\
\check{c}_N &:= \mathbb{E}\big[\boldsymbol{c} \mid \boldsymbol{c} \in -\operatorname{ri} N\big]
\end{aligned}
$$

We replace the continuous cost $\boldsymbol{c}$, by the discrete cost $\check{\boldsymbol{c}}$.

Local and universal exact quantization for $\boldsymbol{c}$

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&= \sum_{N \in \mathcal{N}(P_x)} p_N \check{c}_N{}^\top y_N(x) \\
&= \sum_{N \in \mathcal{N}(P_x)} p_N \min_{y \in P_x} \check{c}_N{}^\top y
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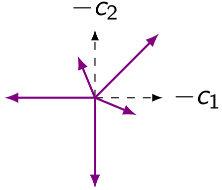

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Contents

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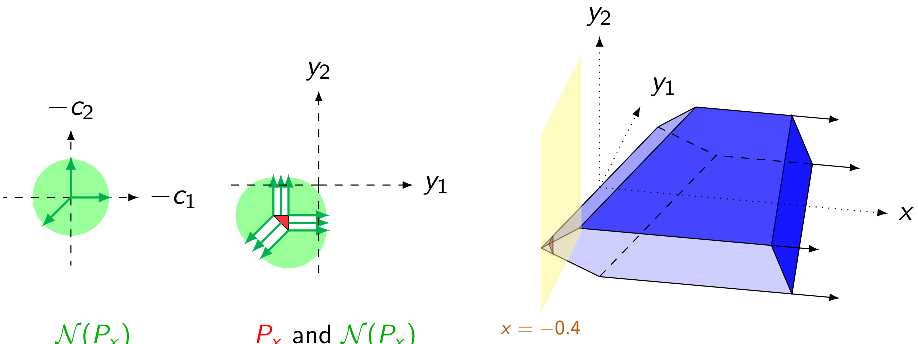

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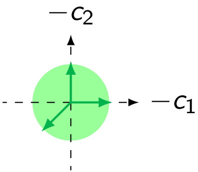

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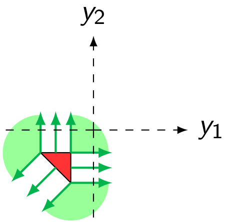

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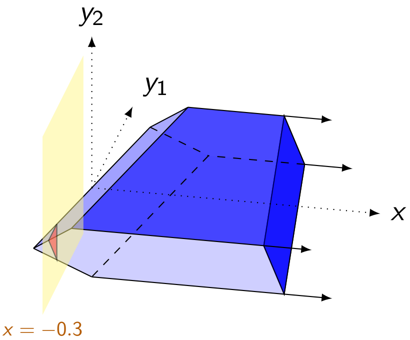

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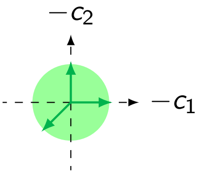

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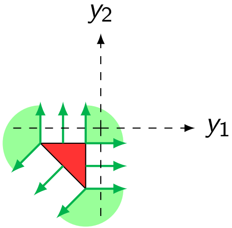

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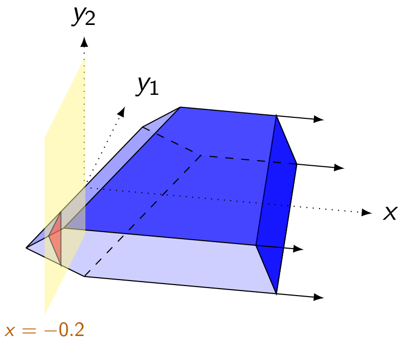

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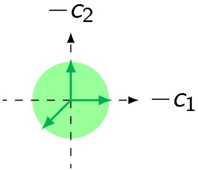

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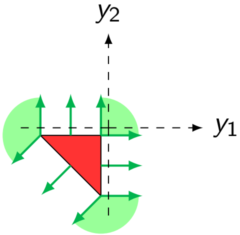

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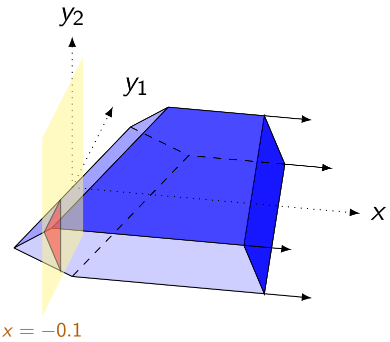

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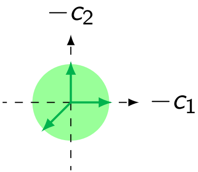

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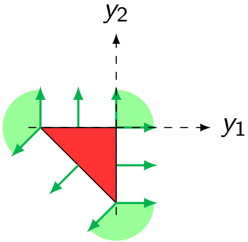

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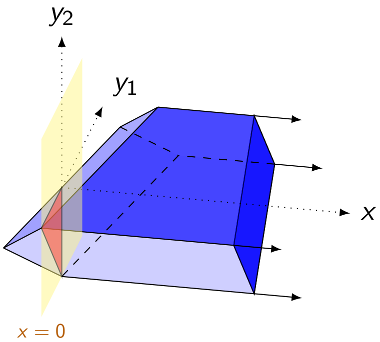

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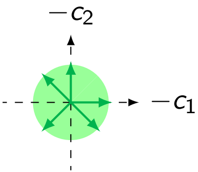

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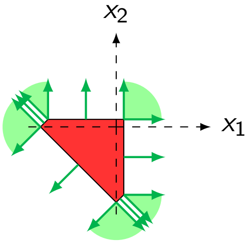
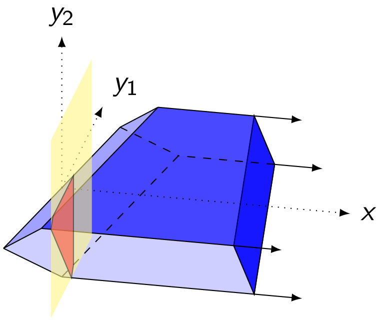


P_x and $\mathcal{N}(P_x)$



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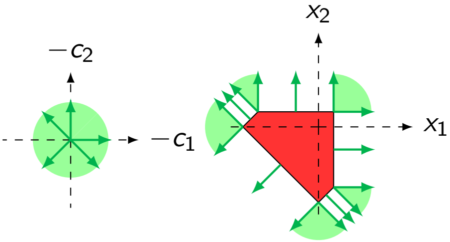
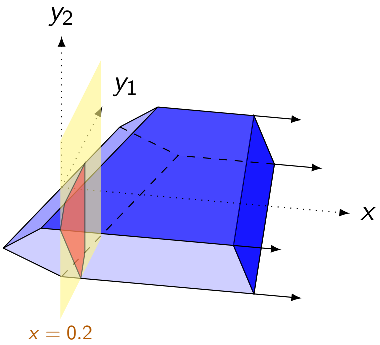


$\mathcal{N}(P_x)$ $\qquad$ P_x and $\mathcal{N}(P_x)$



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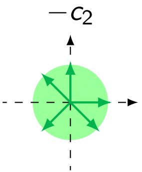

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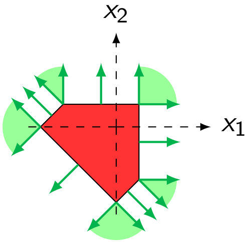

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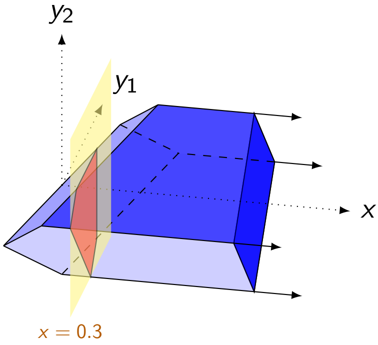

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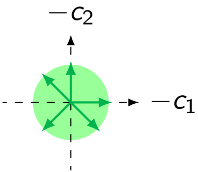

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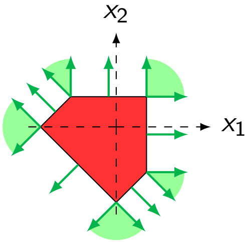

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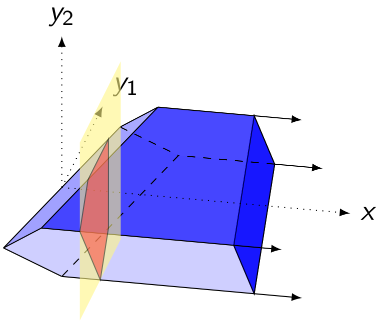

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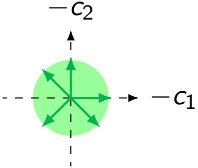

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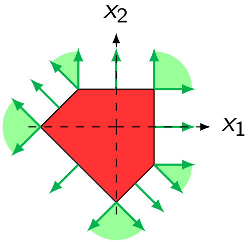

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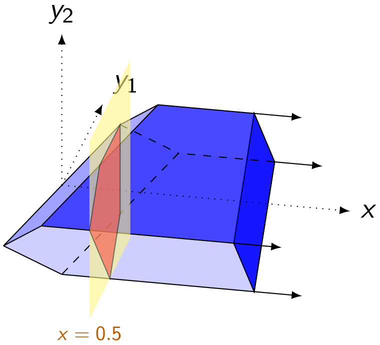

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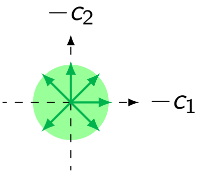

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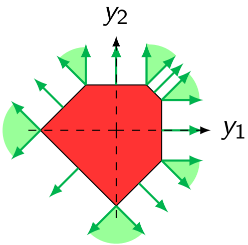

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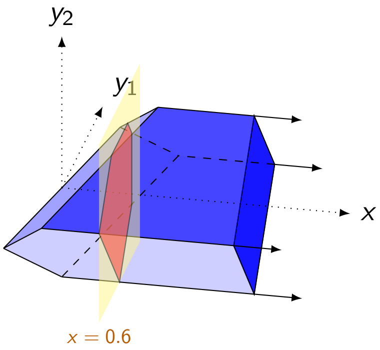

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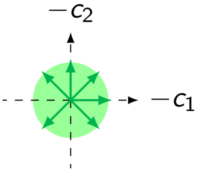

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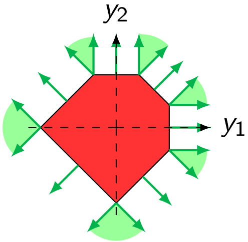

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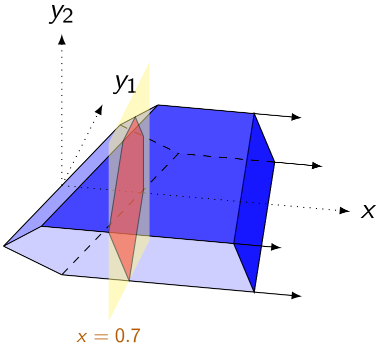

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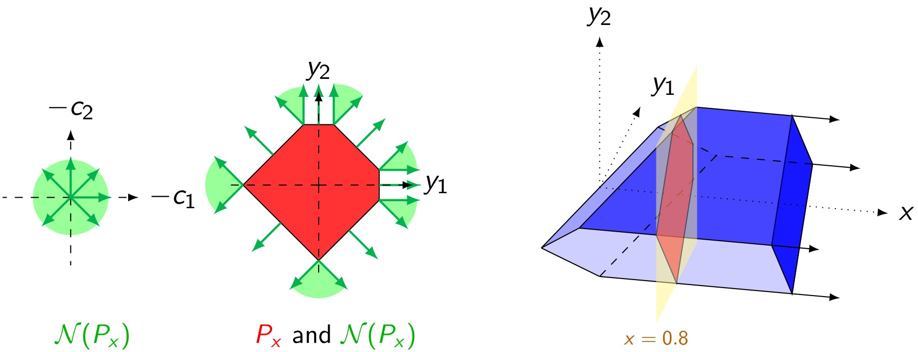

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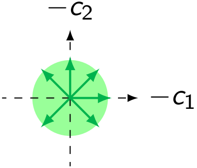

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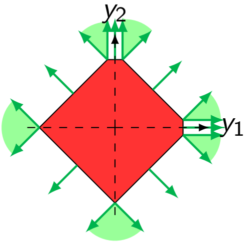

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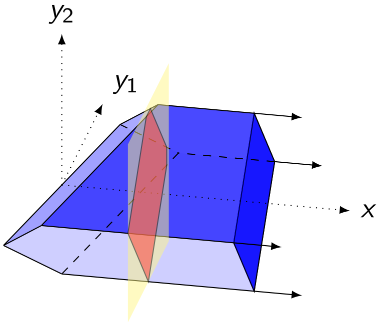

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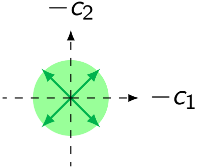

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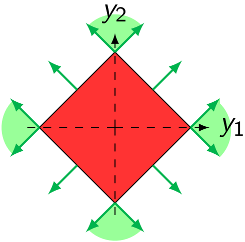

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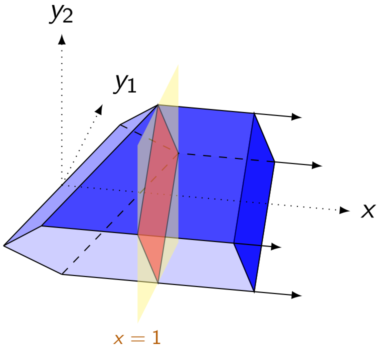

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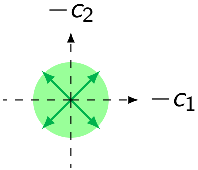

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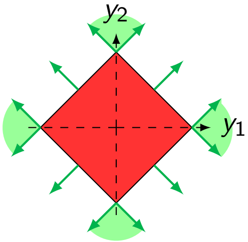

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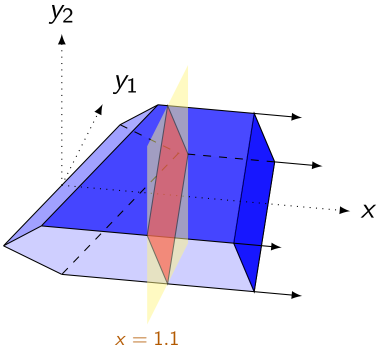

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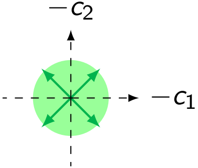

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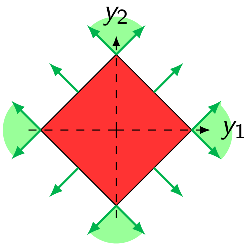

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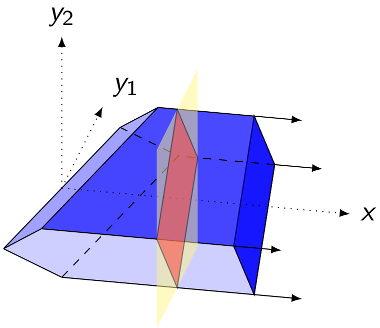

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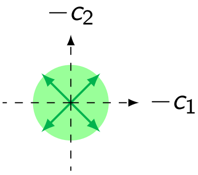

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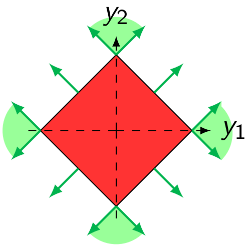

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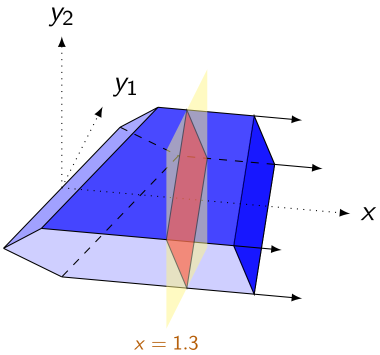

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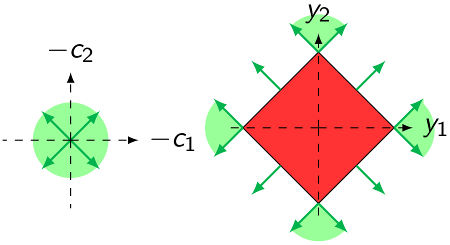

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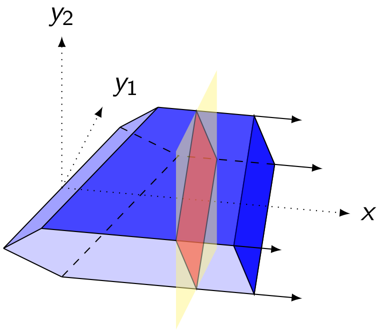

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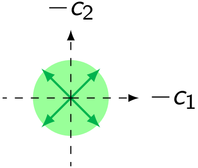

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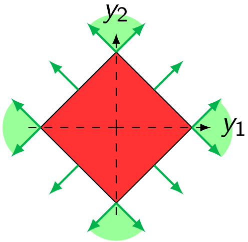

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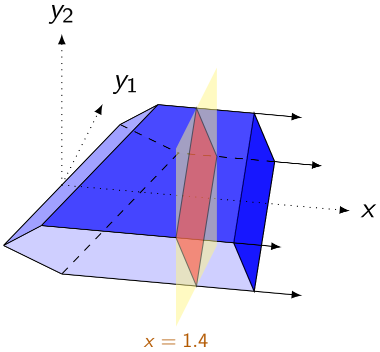

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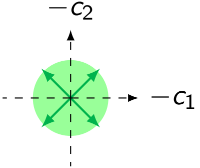

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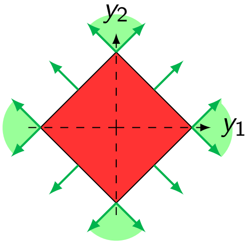

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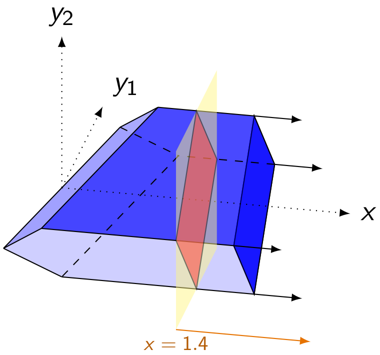

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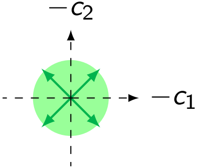

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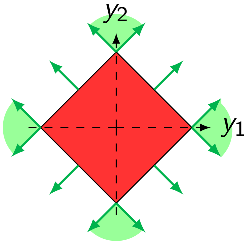

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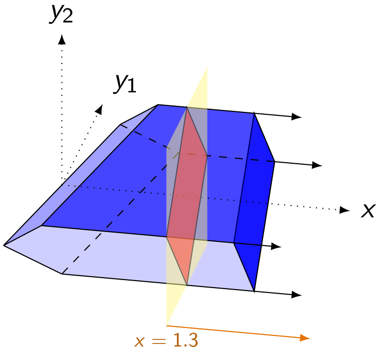

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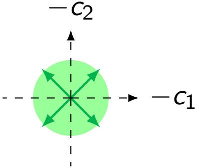

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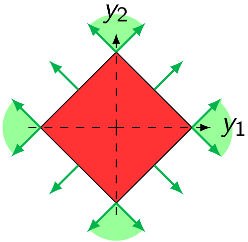

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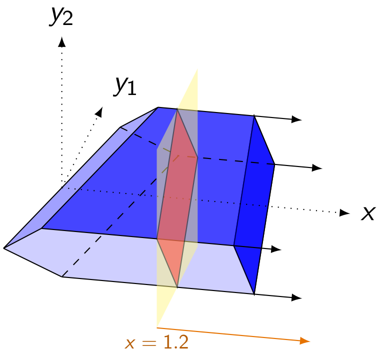

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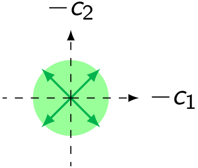

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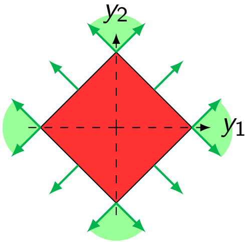

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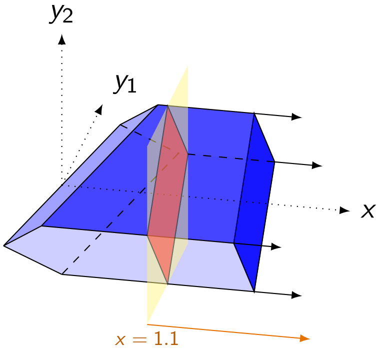

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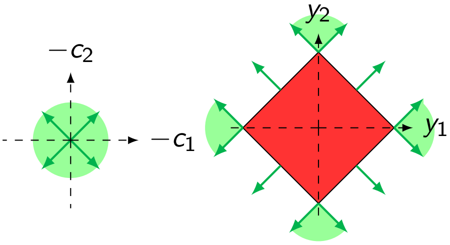

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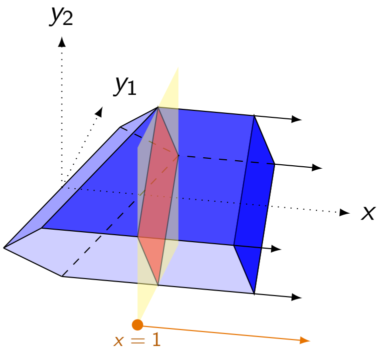

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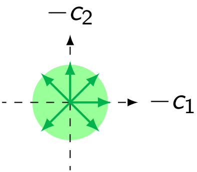

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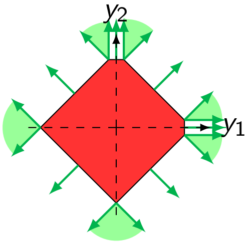

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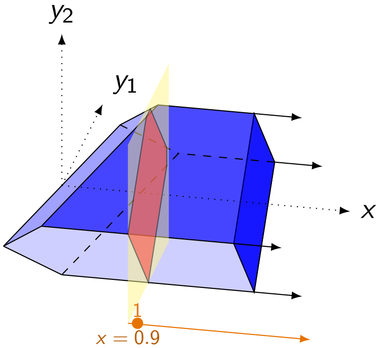

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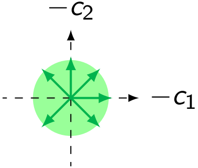
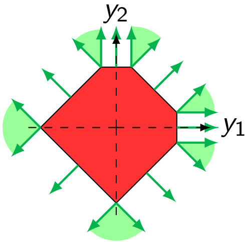
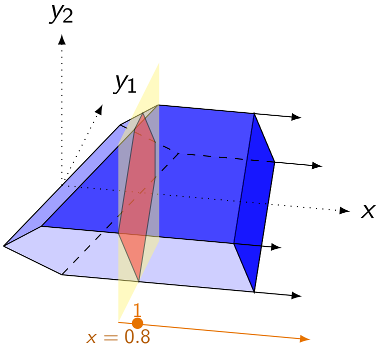


$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$



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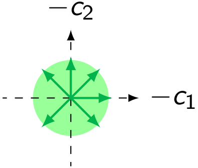

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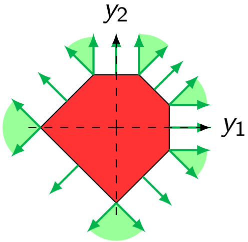

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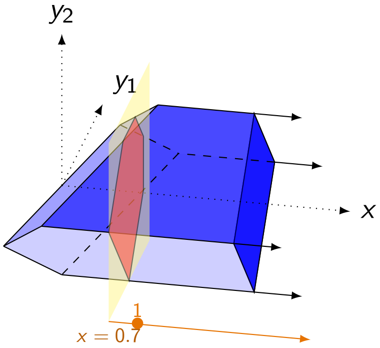

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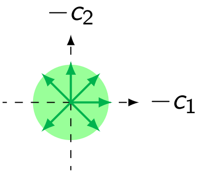

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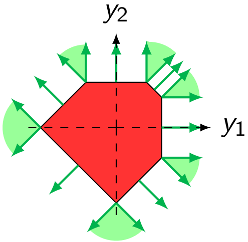

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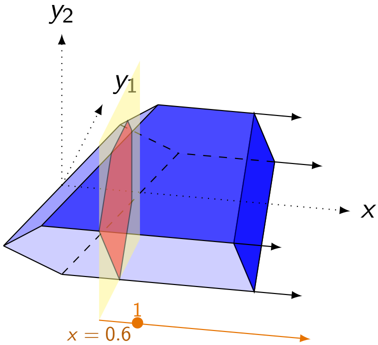

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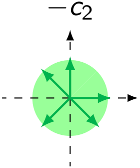

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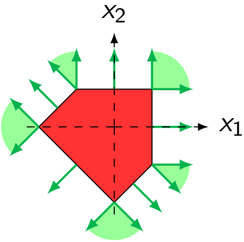

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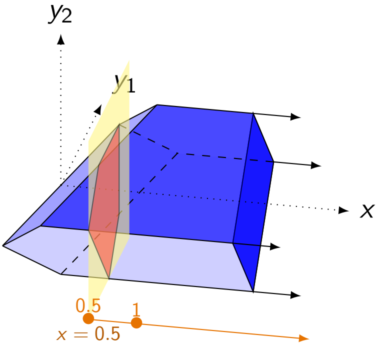

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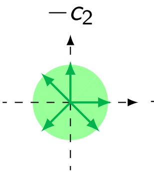

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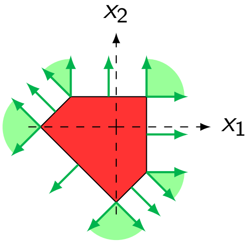

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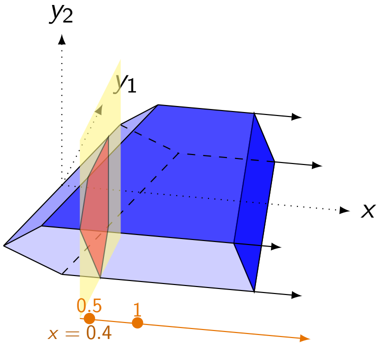

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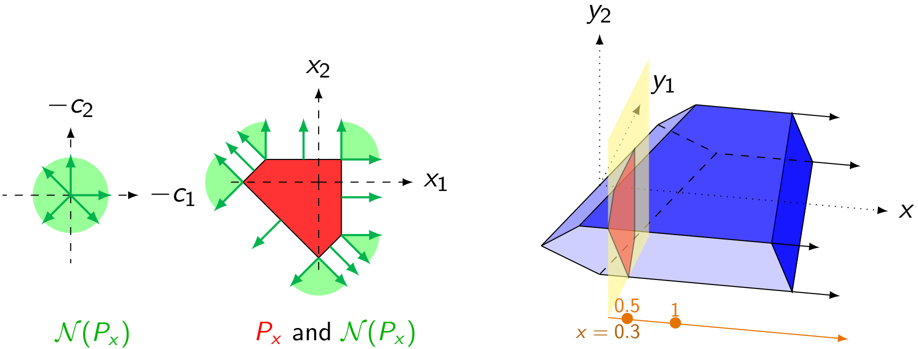

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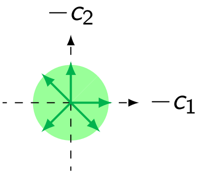

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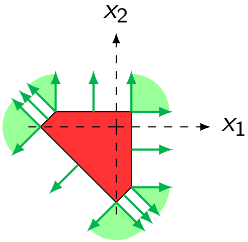

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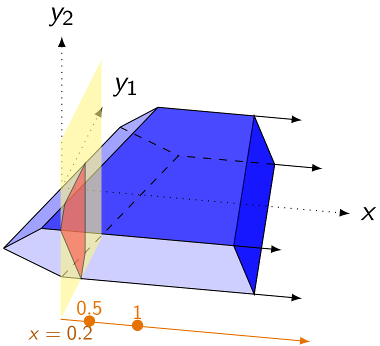

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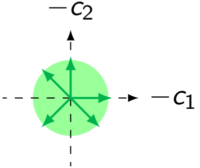

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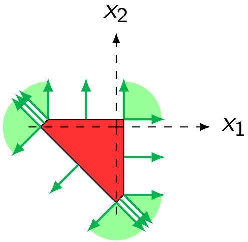

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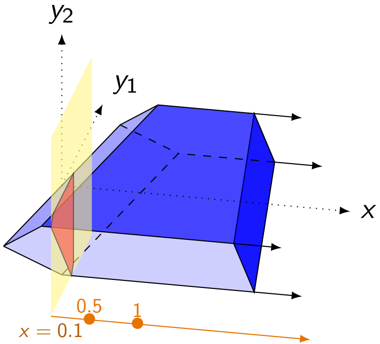

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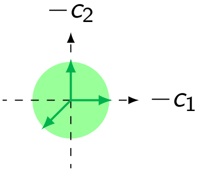

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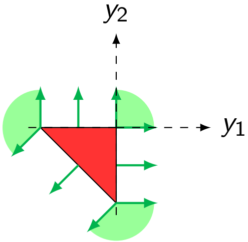

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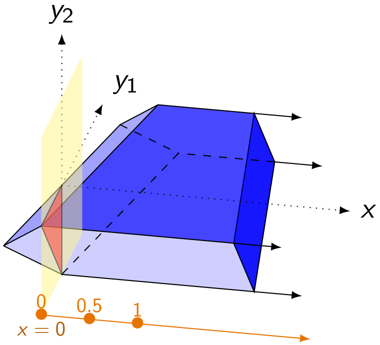

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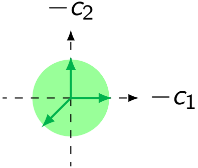

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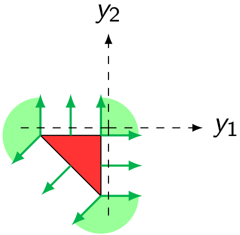

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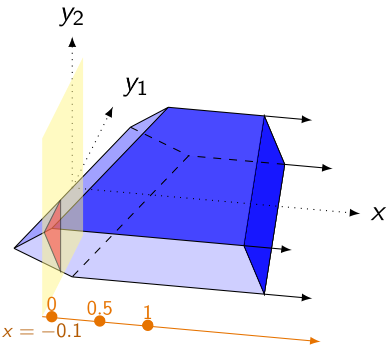

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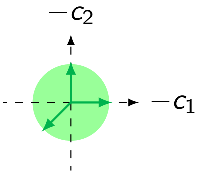

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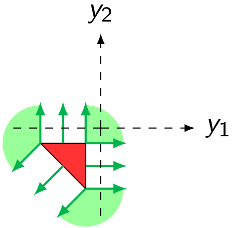

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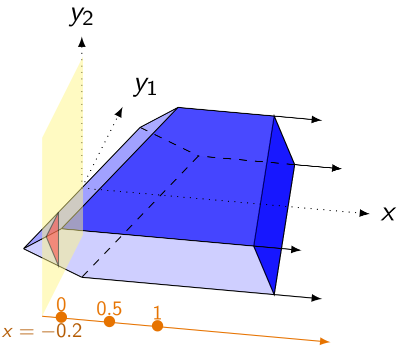

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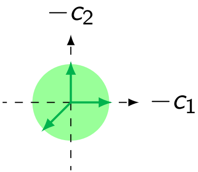

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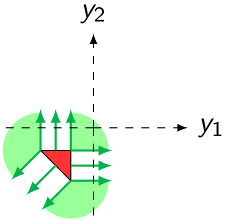

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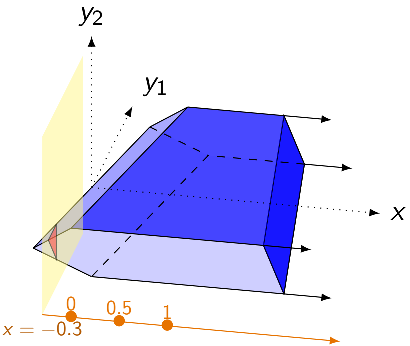

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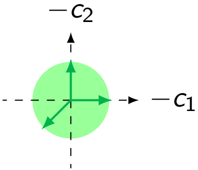

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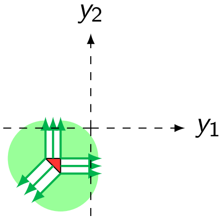

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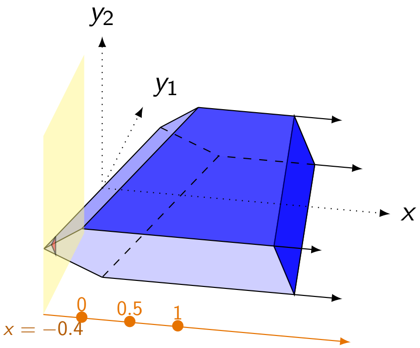


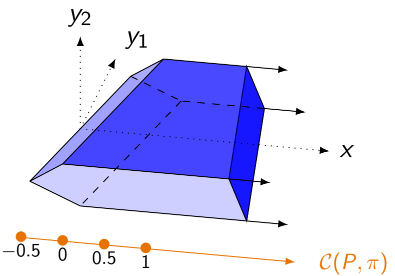
P and P_x

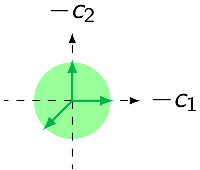
What are the constant regions of $x \mapsto \mathcal{N}(P_x)$?

Proposition

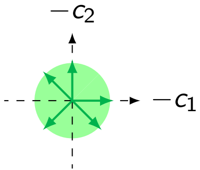
There exists a collection $\mathcal{C}(P,\pi)$ called the **chamber complex** *whose relative interior of cells are the constant regions of $x \mapsto \mathcal{N}(P_x)$.*

I.e, for $\sigma \in \mathcal{C}(P,\pi)$ and $x, x' \in \mathrm{ri}(\sigma)$, we have $\mathcal{N}(P_x) = \mathcal{N}(P_{x'}) =: \mathcal{N}_\sigma$





$\mathcal{N}_\sigma$ for $\sigma = [-0.5, 0]$



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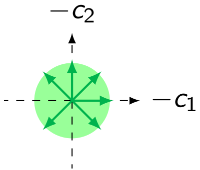
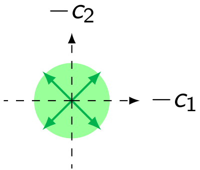


$\mathcal{N}_\sigma$ for $\sigma = [0.5, 1]$



$\mathcal{N}_\sigma$ for $\sigma = [1, +\infty)$

Chamber complex

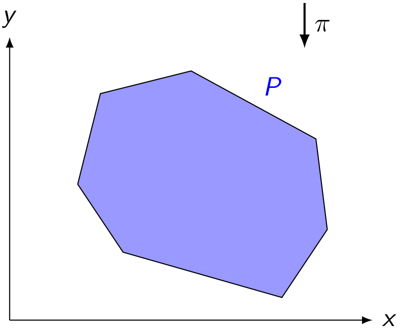
Definition

The *chamber complex* $\mathcal{C}(P, \pi)$ of P along π is

$$\mathcal{C}(P, \pi) := \{\sigma_{P,\pi}(x) \mid x \in \pi(P)\}$$

where

$$\sigma_{P,\pi}(x) := \bigcap_{F \in \mathcal{F}(P) \,|\, x \in \pi(F)} \pi(F)$$



where $\mathcal{F}(P)$ is the set of faces of P and π is the projection $(x, y) \mapsto x$.

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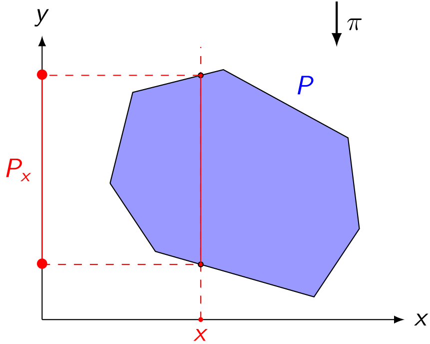

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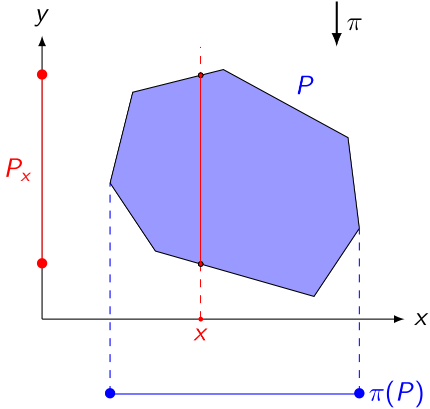

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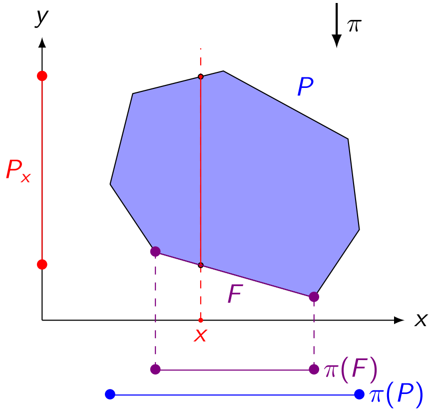

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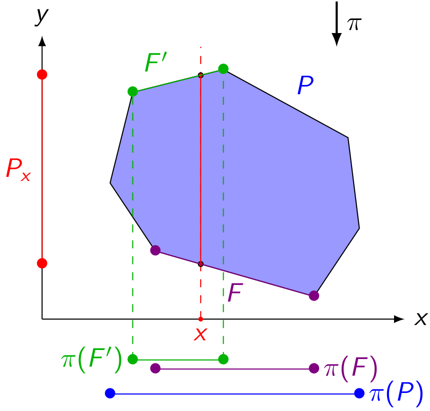

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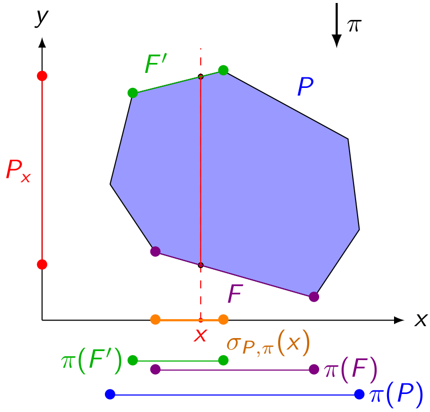

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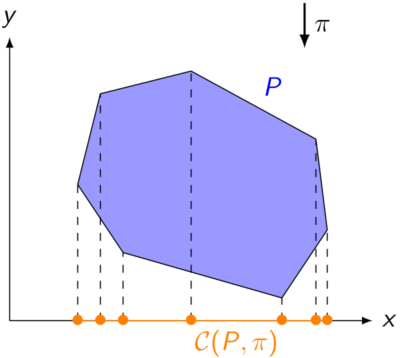

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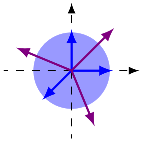
Common Refinement of Normal Fans

We can quantize $\boldsymbol{c}$ on each chamber.



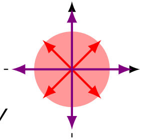
$\mathcal{N}_\sigma$ and $\check{\boldsymbol{c}}$

For all $x \in \mathrm{ri}(\sigma)$,

$$V(x) = \sum_{N \in \mathcal{N}_\sigma} p_N \min_{y \in P_x} \check{c}_N^\top y$$

For all $x' \in \mathrm{ri}(\tau)$,

$$V(x') = \sum_{N \in \mathcal{N}_\tau} p_N \min_{y \in P_x} \check{c}_N^\top y$$



$\mathcal{N}_\tau$ and $\check{\boldsymbol{c}}$

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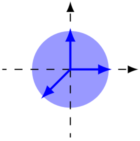

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For all $x' \in \mathrm{ri}(\tau)$,

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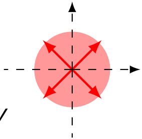
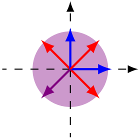


We take the *common refinement*:

$$\mathcal{R} := \mathcal{N}_\sigma \wedge \mathcal{N}_\tau = \{N \cap N' \mid N \in \mathcal{N}_\sigma, N' \in \mathcal{N}_\tau\}$$

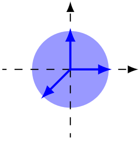


For all $x \in \mathrm{ri}(\sigma) \cup \mathrm{ri}(\tau)$,

$$V(x) = \sum_{N \in \mathcal{N}_\sigma \wedge \mathcal{N}_\tau} p_N \min_{y \in P_x} \check{c}_N^\top y$$

Common Refinement of Normal Fans

We can quantize $\boldsymbol{c}$ on each chamber.



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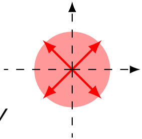

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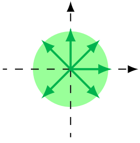

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Uniform exact quantization for c

Let's sum up:

- local exact quantization at x induced by $\mathcal{N}(P_x)$,
- $x \mapsto \mathcal{N}(P_x)$ is constant on each $\sigma \in \mathcal{C}(P, \pi)$,
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Theorem (FGL21, Uniform and universal quantization of the cost)

Let $\mathcal{R} = \bigwedge_{\sigma \in \mathcal{C}(P,\pi)} -\mathcal{N}_\sigma$, *then* **for all** $x \in \mathbb{R}^n$

$$V(x) = \sum_{R \in \mathcal{R}} \check{p}_R \min_{y \in P_x} \check{c}_R^\top y$$

where $\check{p}_R := \mathbb{P}\big[\boldsymbol{c} \in \mathrm{ri}(R)\big]$ *and* $\check{c}_R := \mathbb{E}\big[\boldsymbol{c} \mid \boldsymbol{c} \in \mathrm{ri}(R)\big]$

Polyhedral characterization of V

Theorem (FGL21)

For all distributions of $\boldsymbol{c}$, V is affine on each cell of $\mathcal{C}(P, \pi)$.

Polyhedral characterization of V

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Under an affine change of variable, V is the support function of E

$$V(x) = \sigma_E\big(b - Bx\big) = \sup_{\lambda \in E} (b - Bx)^\top \lambda$$

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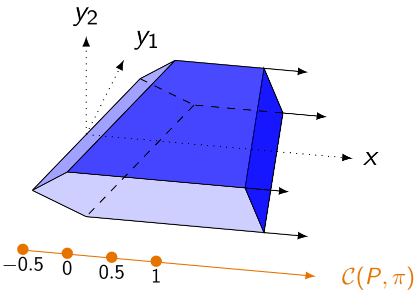
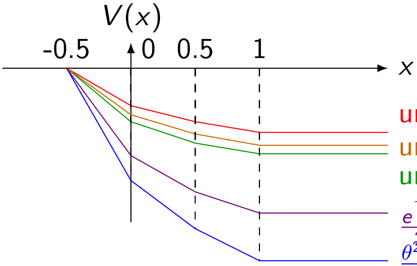
Extension of fiber polytope of

L. Billera, B. Sturmfels, Fiber polytopes, *Annals of Mathematics*, p527–549, 1992.

Explicit computation of the example

$$V(x) = \mathbb{E}\left[\begin{array}{ll} \min_{y\in\mathbb{R}^2} & \boldsymbol{c}^\top y \\ \text{s.t.} & \|y\|_1 \leqslant 1 \\ & y_1 \leqslant x \\ & y_2 \leqslant x \end{array}\right]$$





Different distributions of $\boldsymbol{c}$:

uniform on norm 1 ball

uniform on norm 2 ball

uniform on norm ∞ ball

$\frac{e^{-\frac{\|c\|_2^2}{2\gamma^2}}}{2\pi\gamma^2}dc$

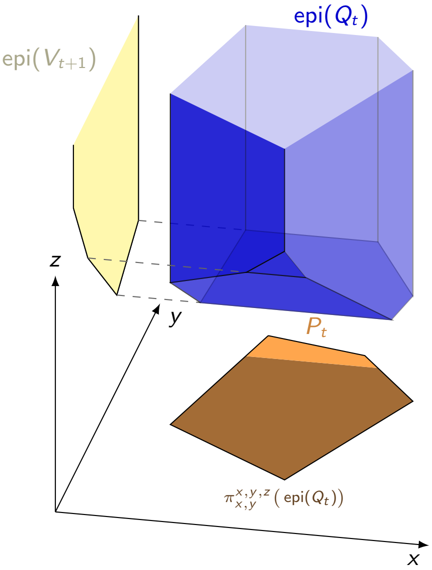
$\frac{\theta^2 e^{-\theta\|c\|_1}}{4}dc$

Contents

Multistage uniform and universal exact quantization

$$V_t(x) = \mathbb{E}\left[\begin{array}{ll} \min\limits_{y \in \mathbb{R}^{n_t}} & \boldsymbol{c}_t^\top y + V_{t+1}(y) \\ & \\ \text{s.t.} & (x,y) \in P_t \end{array}\right]$$

with $Q_t(x,y) := V_{t+1}(y) + \mathbb{I}_{(x,y)\in P_t}$.



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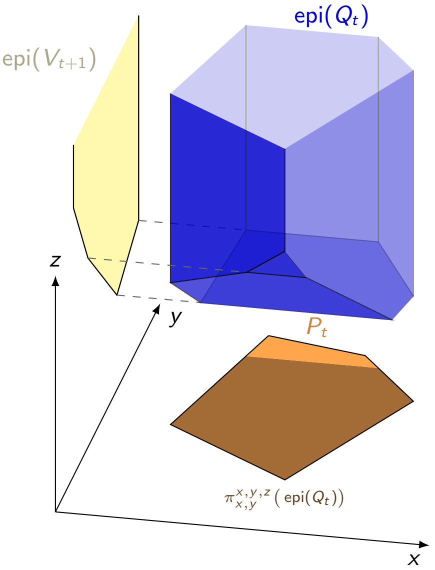

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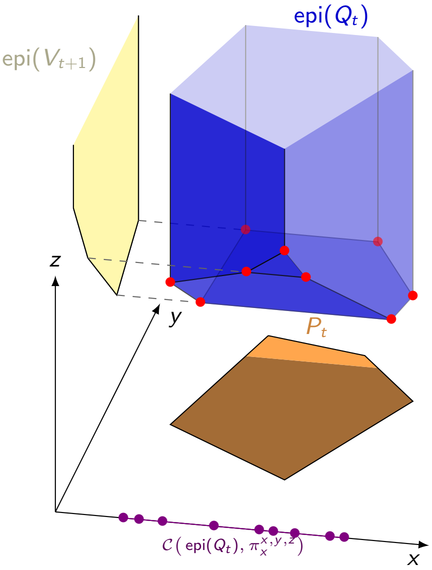

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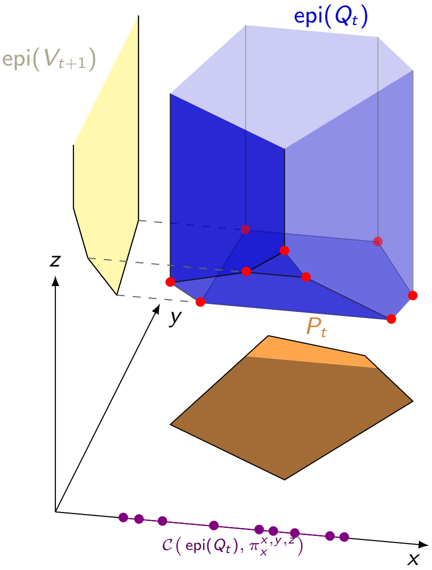

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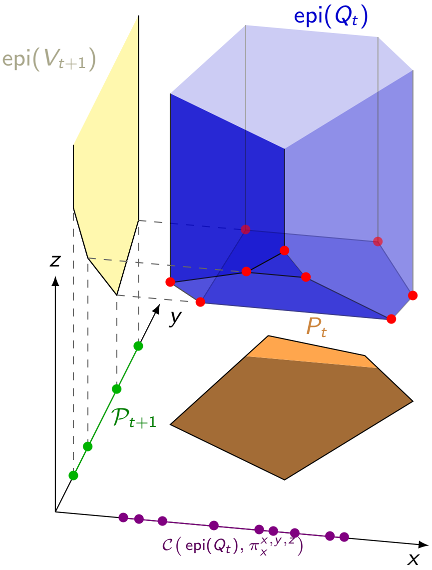

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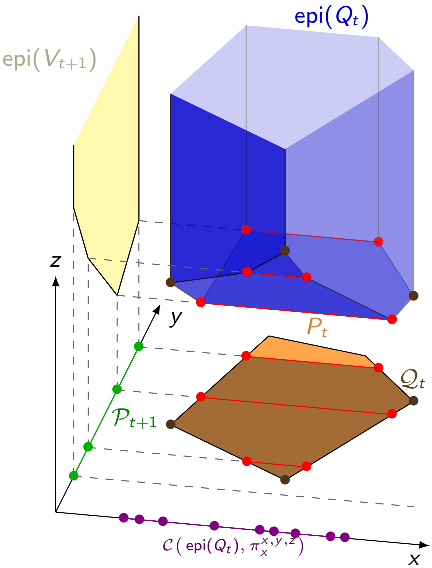

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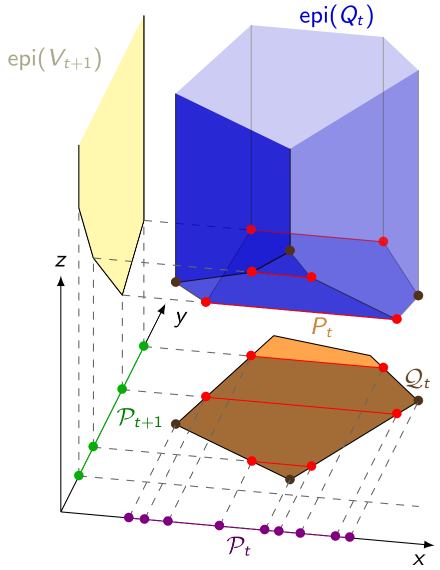

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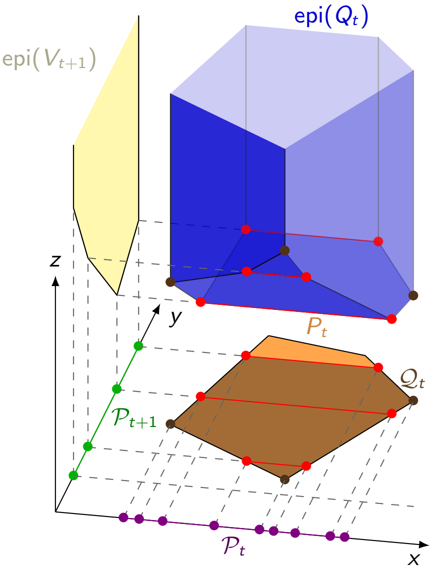

[FGL21, Lem. 4.1]: $\mathcal{P}_t \preccurlyeq \mathcal{C}(\mathrm{epi}(Q_t), \pi_x^{x,y,z})$
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Extension to multistage and stochastic constraints

Iterated chamber complexes by backward induction

$$\mathcal{P}_{t,\xi} := \mathcal{C}\Big(\big(\mathbb{R}^{n_t} \times \mathcal{P}_{t+1}\big) \wedge \mathcal{F}\big(P_t(\xi)\big), \pi_{x_{t-1}}^{x_{t-1},x_t}\Big)$$

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Theorem (FGL21)

All results generalizes to MSLP with finitely supported stochastic constraints.

- $(V_t)_t$ *are affine on universal chamber complexes, i.e. independent of the law of* $(\boldsymbol{c}_t)_t$
- *We have an uniform and universal exact quantization.*

Contents

Earlier and new complexity results

Volume of a polytope

$$\mathrm{Vol}\left(\{z \in \mathbb{R}^d \mid Az \leqslant b\}\right) \text{ or}$$
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- $\sharp P$-complete:
  Dyer and Frieze (1988)
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$$\min_{x \in \mathbb{R}^n} c^\top x + \mathbb{E}\begin{bmatrix} \min\limits_{y \in \mathbb{R}^{m}} \boldsymbol{q}^\top y \\ \text{s.t. } Tx + Wy \leqslant h \end{bmatrix}$$
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Complexity result multistage

Theorem (FGL21: MSLP is polynomial for fixed dimensions)

Assume that T, $n_2, \cdots, n_T$, are fixed.[1]
Assume that $\boldsymbol{c}$ admits a density function with a bounded total variation.

Then, there exists an algorithm that either asserts that MSLP is unfeasible or finds an ε-solution in polynomial time in $\log(\frac{1}{\varepsilon})$ with probability 1.

[1] No requirement for the first decision.

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Same with SDDP: [Lan 2020][Zhang and Sun 2020]

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2 stage stochastic linear programming (2SLP)

$$\min_{x \in \mathbb{R}^n_+} \quad c^\top x + \mathbb{E}\big[Q(x, \boldsymbol{\xi})\big]$$
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where $\boldsymbol{\xi} = (\boldsymbol{T}, \boldsymbol{h})$ is random whereas q and W are deterministic[1]

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$$\begin{aligned} \min_{x \in \mathbb{R}^n_+} \quad & c^\top x + \mathbb{E}\big[Q(x, \boldsymbol{\xi})\big] \\ \text{s.t.} \quad & Ax = b \end{aligned}$$

where $\boldsymbol{\xi} = (\boldsymbol{T}, \boldsymbol{h})$ is random whereas q and W are deterministic[1]

$$Q(x, \xi) := \min_{y \in \mathbb{R}^m_+} q^\top y \quad \text{s.t.} \quad Tx + Wy = h \qquad = \max_{\lambda \in \mathbb{R}^n} (h - Tx)^\top \lambda \quad \text{s.t.} \quad W^\top \lambda \leqslant q$$

We define

$$X := \{x \in \mathbb{R}^n_+ \mid Ax = b\} \qquad D := \{\lambda \in \mathbb{R}^l \mid W^\top \lambda \leqslant q\}$$

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$\rightsquigarrow$ need to discretize $\boldsymbol{\xi}$

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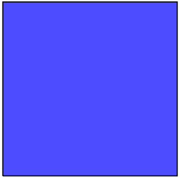
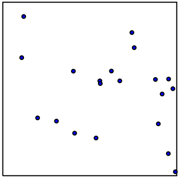
Partitioning the cost-to-go function



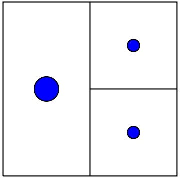
$\boldsymbol{\xi}$ continuous

$$V(x) = \mathbb{E}\left[Q\left(x, \boldsymbol{\xi}\right)\right]$$



SAA $N = 20$

$$V_N^{SAA}(x) = \frac{1}{N}\sum_{k=1}^{N} Q\left(x, \xi^k\right)$$



Partition-based

$$V_{\mathcal{P}}(x)$$

Definition (Partitioned expected-cost-go)

Let $\mathcal{P}$ be a $\mathbb{P}$-partition of Ξ, we define

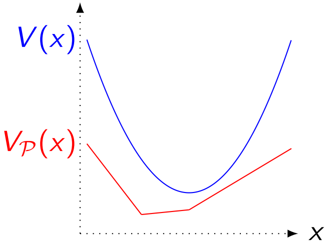
$$V_{\mathcal{P}}(x) := \sum_{P \in \mathcal{P}} \mathbb{P}\left[P\right] Q\left(x, \mathbb{E}\left[\boldsymbol{\xi} | P\right]\right)$$

Properties of partitioned cost-to-go

Recall that

$$V(x) = \mathbb{E}\Big[Q(x,\xi)\Big]$$
$$V_{\mathcal{P}}(x) = \sum_{P\in\mathcal{P}} \mathbb{P}\big[P\big]\, Q\big(x, \mathbb{E}\big[\xi|P\big]\big)$$

- $Q(x,\cdot)$ is convex $\rightsquigarrow V_{\mathcal{P}} \leqslant V$.
- $Q\big(\cdot,\mathbb{E}\big[\xi|P\big]\big)$ is polyhedral $\rightsquigarrow V_{\mathcal{P}}$ is polyhedral.



Finally,

$$\min_{x\in X} c^\top x + V_{\mathcal{P}}(x) \qquad (2SLP_{\mathcal{P}})$$

is equivalent to

$$\begin{aligned}\min_{x\in X,(y_P)_{P\in\mathcal{P}}} \quad & c^\top x + \sum_{P\in\mathcal{P}} \mathbb{P}\big[P\big] q^\top y_P \\ & \mathbb{E}\big[\boldsymbol{T}|P\big]x + Wy_P \leqslant \mathbb{E}\big[\boldsymbol{h}|P\big] \qquad \forall P\in\mathcal{P}\end{aligned}$$

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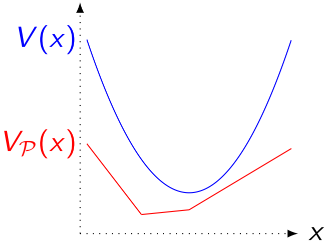

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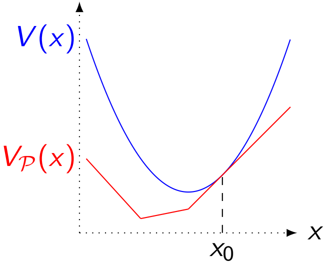
Adapted partition

Definition

We say that a partition $\mathcal{P}$ is adapted to x_0 if

$$V_{\mathcal{P}}(x_0) = V(x_0) := \mathbb{E}\big[Q(x_0,\xi)\big]$$



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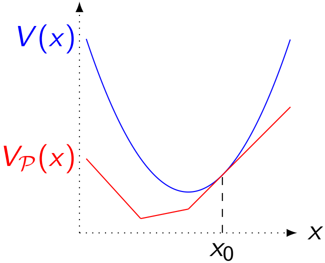

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An partition oracle is a function taking a first stage decision x^k as argument and returning an partition of Ξ.

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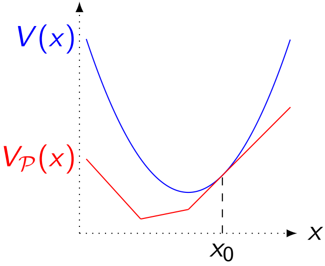

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An adapted partition oracle is a function taking a first stage decision x^k as argument and returning an adapted to x^k partition of Ξ.

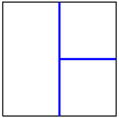
Refinement

$\mathcal{R}$ refines $\mathcal{P}$ ($\mathcal{R} \preccurlyeq \mathcal{P}$) if

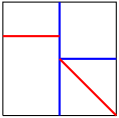
$$\forall R \in \mathcal{R}, \exists P \in P, R \subset P$$

[$\mathcal{R} \preccurlyeq_{\mathbb{P}} \mathcal{P}$ if $\mathcal{R}$ refines $\mathcal{P}$ up to $\mathbb{P}$-null sets.]

Then, $\quad \mathcal{R} \preccurlyeq_{\mathbb{P}} \mathcal{P} \Rightarrow V_{\mathcal{R}} \geqslant V_{\mathcal{P}}$



$\mathcal{P}$



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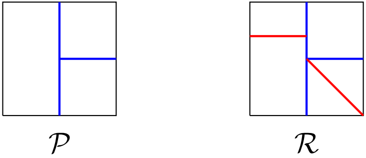
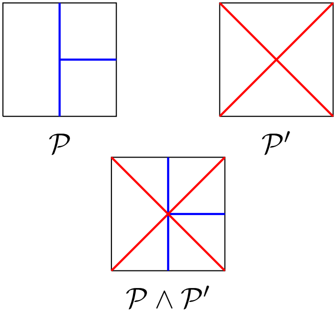


The common refinement of $\mathcal{P}$ and $\mathcal{P}'$ is

$$\mathcal{P} \wedge \mathcal{P}' := \{P \cap P' \,|\, P \in \mathcal{P}, P' \in \mathcal{P}'\}$$

Since $\mathcal{P} \wedge \mathcal{P}'$ refines $\mathcal{P}$ and $\mathcal{P}'$

$$\max(V_{\mathcal{P}}, V_{\mathcal{P}'}) \leqslant V_{\mathcal{P} \wedge \mathcal{P}'}$$



General framework for APM

$k \leftarrow 0$, $z_U^0 \leftarrow +\infty$, $z_L^0 \leftarrow -\infty$, $\mathcal{P}^0 \leftarrow \{\Xi\}$;

while $z_U^k - z_L^k > \varepsilon$ **do**

- $k \leftarrow k+1$;
- Solve (for x^k) $\quad z_L^k \leftarrow \min_{x \in X} c^\top x + V_{\mathcal{P}^{k-1}}(x)$;
- $\mathcal{P}_{x^k} \leftarrow \mathrm{Oracle}(x^k)$;
- $\mathcal{P}^k \leftarrow \mathcal{P}^{k-1} \wedge \mathcal{P}_{x^k}$;
- $z_U^k \leftarrow \min\left(z_U^{k-1}, c^\top x^k + V_{\mathcal{P}^k}(x^k)\right)$;

end

Algorithm 1: Generic framework for APM.

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Algorithm 1: Generic framework for APM.

Theorem (FL2021)

If the oracle is adapted, then x^k is an ε-solution of problem (2SLP) for $k \geqslant \left(\frac{L diam(X)}{\varepsilon} + 1\right)^n$.

Previous APM methods

Lemma (Song & Luedtke)

Let $\mathcal{P}$ a partition of Ξ. $\mathcal{P}$ is adapted at x iff for all set of scenarios $P \in \mathcal{P}$, there exists a common optimal multiplier λ_P, i.e.

$$\forall P \in \mathcal{P}, \quad \exists \lambda_P \in D, \quad \forall \xi_k \in P, \qquad \lambda_P \in \operatorname*{argmax}_{\lambda \in D} (h^k - T^k x)^\top \lambda$$

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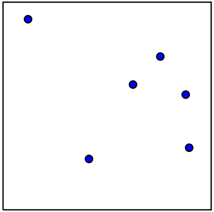
Idea

- Sample a large number of scenario
- without loss of precision aggregate scenarios



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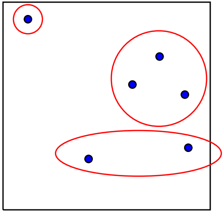

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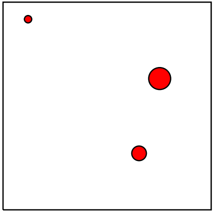

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Lemma (Ramirez-Pico & Moreno)

Let $\mathcal{P}$ a partition of Ξ. If there exists $\lambda(\boldsymbol{\xi})$ such that, for all $P \in \mathcal{P}$,

$$\mathbb{E}\big[\boldsymbol{h}|P\big]^\top \mathbb{E}\big[\lambda(\boldsymbol{\xi})|P\big] = \mathbb{E}\big[\boldsymbol{h}^\top \lambda(\boldsymbol{\xi})|P\big]$$
$$x^\top \mathbb{E}\big[\boldsymbol{T}|P\big]^\top \mathbb{E}\big[\lambda(\boldsymbol{\xi})|P\big] = x^\top \mathbb{E}\big[\boldsymbol{T}^\top \lambda(\boldsymbol{\xi})|P\big]$$

then $\mathcal{P}$ is an adapted partition.

A (partial) comparison between partition based results

Paper	Song, Luedtke (2015)	Ramirez-Pico, Moreno (2020)	Forcier, L. (2021)
Non-finite supp(ξ)	×	✓	✓
Explicit oracle	✓	×	✓
Proof of convergence	✓	×	✓
Complexity result	×	×	✓
Fast iteration	✓	×	×

Local exact quantization and adapted partition

Local exact quantization

random cost

Recall that for a fixed x,

$$\mathbb{E}\Big[\min_{y \in P_x} \boldsymbol{c}^\top y\Big] = \sum_{N \in \mathcal{N}(P_x)} p_N \min_{y \in P_x} \check{c}_N^\top y$$

where,

$$p_N := \mathbb{P}\big[\boldsymbol{c} \in -\operatorname{ri} N\big]$$
$$\check{c}_N := \mathbb{E}\big[\boldsymbol{c} \mid \boldsymbol{c} \in -\operatorname{ri} N\big]$$

$$P_x := \{y \in \mathbb{R}^m \mid Ay + Bx \leqslant b\}$$

GAPM

random constraints

Similarly, for a given q, and all x,

$$\begin{aligned} V(x) &:= \mathbb{E}\big[Q(x, \boldsymbol{\xi})\big] \\ &= \mathbb{E}\Big[\max_{\lambda \in D_q} (\boldsymbol{h} - \boldsymbol{T}x)^\top \lambda\Big] \\ &= \sum_{N \in \mathcal{N}(D_q)} p_N \max_{\lambda \in D_q} \psi_{N,x}^\top \lambda \end{aligned}$$

where,

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$$\psi_{N,x} := \mathbb{E}\big[\boldsymbol{h} - \boldsymbol{T}x \mid \boldsymbol{h} - \boldsymbol{T}x \in \operatorname{ri} N\big]$$

$$D_q := \{\lambda \in \mathbb{R}^l \mid W^\top \lambda \leqslant q\}$$

An explicit adapted partition

Consider $x \in \mathbb{R}^n$ and $N \in \mathcal{N}(D_q)$ a normal cone of D_q. We define

$$E_{N,x} := \{\xi \in \Xi \mid h - Tx \in \operatorname{ri} N\}$$

Theorem (FL 2021)

$\mathcal{R}_x := \left\{E_{N,x} \mid N \in \mathcal{N}(D_q)\right\}$ *is an adapted partition to* x
i.e. $V_{\mathcal{R}_x}(x) = V(x)$

Proof:

$$\begin{aligned} V(x) &:= \mathbb{E}\left[Q(x,\xi)\right] \\ &= \sum_{N \in \mathcal{N}(D)} \mathbb{P}\left[\boldsymbol{h} - \boldsymbol{T}x \in \operatorname{ri} N\right] \min_{\lambda \in D} \mathbb{E}\left[\boldsymbol{h} - \boldsymbol{T}x \,|\, \boldsymbol{h} - \boldsymbol{T}x \in \operatorname{ri} N\right]^\top \lambda \\ &= \sum_{N \in \mathcal{N}(D)} \mathbb{P}\left[\boldsymbol{\xi} \in E_{N,x}\right] Q\Big(\mathbb{E}\left[\boldsymbol{\xi} \,|\, \boldsymbol{\xi} \in E_{N,x}\right], x\Big) = V_{\mathcal{R}_x}(x) \end{aligned}$$

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\end{aligned}$$

➥ Is it the coarsest one ?

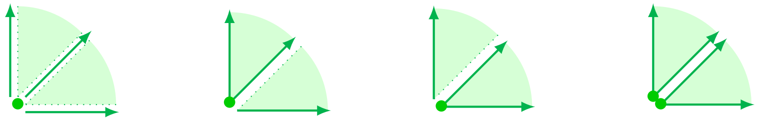
CNS conditions for a partition to be adapted

Theorem (FL 2021)

For $x \in \mathbb{R}^n$ and $\mathcal{P}$ a partition of Ξ, there exists $\overline{\mathcal{R}}_x \succcurlyeq_{\mathbb{P}} \mathcal{R}_x$ such that

$$\mathcal{P} \preccurlyeq_{\mathbb{P}} \overline{\mathcal{R}}_x \iff V_{\mathcal{P}}(x) = V(x).$$

- If $\boldsymbol{\xi}$ admits a density, $\mathcal{R}_x =_{\mathbb{P}} \overline{\mathcal{R}}_x$.
- An oracle is adapted if and only if it returns a partition $\mathcal{P}$ refining $\overline{\mathcal{R}}_x$.



$\mathcal{R}_x$ $\mathcal{P}$ $\mathcal{P}'$ $\overline{\mathcal{R}}_x$

$$E_{N,x} := \{\xi \in \Xi \mid h - Tx \in \mathrm{ri}(N)\}$$
$$\mathcal{R}_x := \left\{E_{N,x} \mid N \in \mathcal{N}(D_q)\right\}$$

$$\overline{E}_{N,x} := \{\xi \in \Xi \mid h - Tx \in N\}$$
$$\overline{\mathcal{R}}_x := \left\{E_{N,x} \mid N \in \mathcal{N}(D_q)^{\max}\right\}.$$

Stochastic cost and recourse

- We have shown a local exact quantization result for random $\boldsymbol{T}, \boldsymbol{h}$, and deterministic q, W.
- If $\boldsymbol{q}$ and $\boldsymbol{W}$ are finitely supported random variable:
  1. compute an exact quantization $\mathcal{N}_\xi$ for every element of the support;
  2. take the common refinement.

We have seen that we can deal with non-finitely supported $\boldsymbol{q}$ through the chamber complexes.

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Adapted partition for general q

We define coupling constraint and fiber for the dual.

$$\begin{aligned} D_q &:= \left\{ \lambda \in \mathbb{R}^\ell \quad \middle| \quad W^\top \lambda \leqslant q \right\} \\ \Delta &:= \left\{ (\lambda, q) \in \mathbb{R}^\ell \times \mathbb{R}^m \quad \middle| \quad W^\top \lambda \leqslant q \right\} \\ \mathcal{R}_{x,q} &:= \left\{ E_{N,x} \quad \middle| \quad N \in \mathcal{N}(D_q) \right\} \end{aligned}$$

Recall that $q \mapsto \mathcal{N}(D_q)$ is piecewise constant on $\mathcal{C}(\Delta, \pi_\lambda^{\lambda,q})$ and so is $\mathcal{R}_{x,q}$.
➡ we can take the common refinement of a finite number of $\mathcal{R}_{x,q}$!!

More precisely:

- The chamber complex $\mathcal{C}(\Delta, \pi_\lambda^{\lambda,q}) = \Sigma\text{-fan}(W)^2$.
- For $S \in \Sigma\text{-fan}(W)$ define $\mathcal{R}_{x,S} := \mathcal{R}_{x,q}$ for any $q \in \mathrm{ri}(S)$.

➡ $\left\{ \mathrm{ri}(S) \times R \,\middle|\, S \in \Sigma\text{-fan}(W), R \in \mathcal{R}_{x,S} \right\}$ is an adapted partition to x.

[2]The well studied secondary fan of W

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Synthesis of local and uniform quantization results

	$\boldsymbol{W}$	$(\boldsymbol{T}, \boldsymbol{h})$	$\boldsymbol{q}$
Local	$\emptyset$	$\mathcal{R}_x$	$\mathcal{N}(P_x)$
Uniform	$\emptyset$	$\emptyset$	$\bigwedge_{\sigma \in \mathcal{C}(P,\pi)} \mathcal{N}_\sigma$

Subgradient of partition function

Recall that if $\mathcal{P} \preccurlyeq_{\mathbb{P}} \mathcal{R}_x$ then

$$V_{\mathcal{R}_x}(x) = V_{\mathcal{P}}(x) = V(x)$$

$$V_{\mathcal{R}_x}(\cdot) \leqslant V_{\mathcal{P}}(\cdot) \leqslant V(\cdot)$$

Lemma

Let $x \in \operatorname{dom}(V)$ *and* $\mathcal{P}$ *be a refinement of* $\mathcal{R}_x$*, i.e.* $\mathcal{P} \preccurlyeq \mathcal{R}_x$*, then*

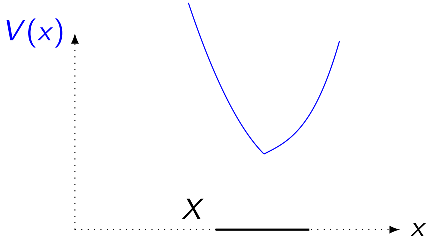
$$\partial V_{\mathcal{R}_x}(x) \subset \partial V_{\mathcal{P}}(x) \subset \partial V(x)$$

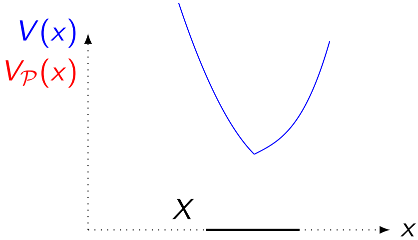
Furthermore, if $x \in \operatorname{ri}\operatorname{dom}(V)$*,*

$$\partial V_{\mathcal{R}_x}(x) = \partial V_{\mathcal{P}}(x) = \partial V(x)$$

Link with Benders decomposition and L-shaped

Partition based method can be seen as a tangent cone method: a cutting plane method where we add all active cuts instead of a single one.





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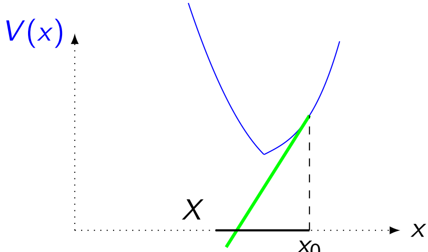

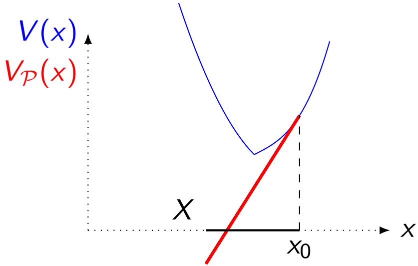

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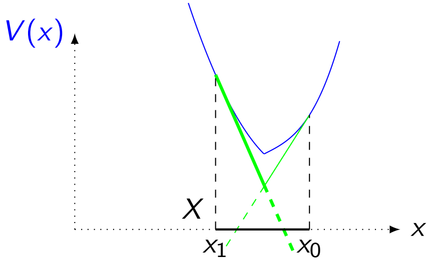

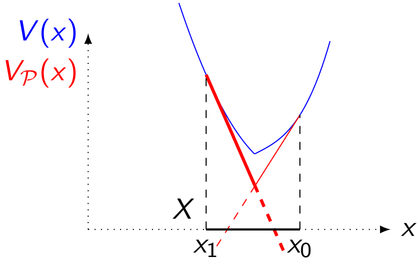

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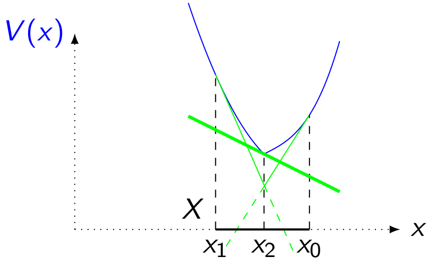

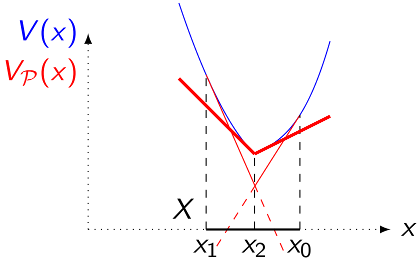

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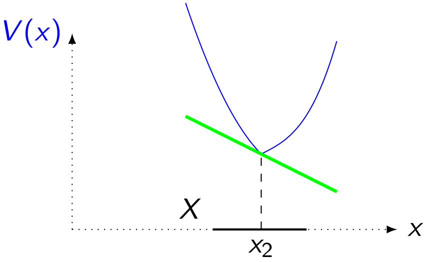

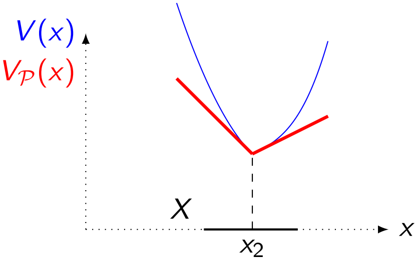

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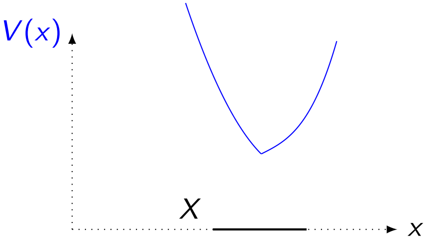

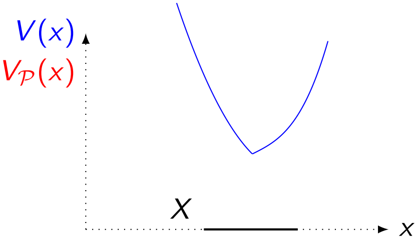

Theorem (Convergence and complexity results)

If $X \cap \mathrm{dom}(V) \subset \mathbb{R}^+$ is contained in a ball of diameter $M \in \mathbb{R}^+$ and $x \to c^\top x + V(x)$ is Lipschitz with constant L then the partition based method finds an ε-solution in at most $\left(\frac{LM}{\varepsilon} + 1\right)^n$ iterations.

Explicit formulas for usual distributions

Recall that $V_{\mathcal{P}}(x) = \sum_{P\in\mathcal{P}} \mathbb{P}\big[P\big] Q\big(x, \mathbb{E}\big[\boldsymbol{\xi}|P\big]\big)$.

Thus, we need to compute $\mathbb{P}\big[C\big]$ and $\mathbb{E}\big[\boldsymbol{\xi} \mid C\big]$ when C is a polyhedron.

Fortunately we have some explicit formulas, valid for S full dimensional simplex or simplicial cone, which can be used through triangulation.

Distribution	Uniform on polytope	Exponential	Gaussian
$d\mathbb{P}(\xi)$	$\frac{\mathbb{1}_{\xi\in Q}}{\mathrm{Vol}_d(Q)}\mathcal{L}_{\mathrm{Aff}(Q)}(d\xi)$	$\frac{e^{\theta^\top\xi}\mathbb{1}_{\xi\in K}}{\Phi_K(\theta)}\mathcal{L}_{\mathrm{Aff}(K)}(d\xi)$	$\frac{e^{-\frac{1}{2}\xi^\top M^{-2}\xi}}{(2\pi)^{\frac{m}{2}}\det M}d\xi$
Support	Polytope : Q	Cone : K	$\mathbb{R}^m$
$\mathbb{P}\big[S\big]$	$\dfrac{\mathrm{Vol}_d(S)}{\mathrm{Vol}_d(Q)}$	$\dfrac{\lvert\det(\mathrm{Ray}(S))\rvert}{\Phi_K(\theta)}\prod_{r\in\mathrm{Ray}(S)}\dfrac{1}{-r^\top\theta}$	$\mathrm{Ang}\left(M^{-1}S\right)$
$\mathbb{E}\big[\xi \mid S\big]$	$\frac{1}{d}\sum_{v\in\mathrm{Vert}(S)} v$	$\left(\sum_{r\in\mathrm{Ray}(S)}\frac{-r_i}{r^\top\theta}\right)_{i\in[m]}$	$\frac{\sqrt{2}\Gamma(\frac{m+1}{2})}{\Gamma(\frac{m}{2})}M\,\mathrm{Ctr}\left(S\cap\mathbb{S}_{m-1}\right)$

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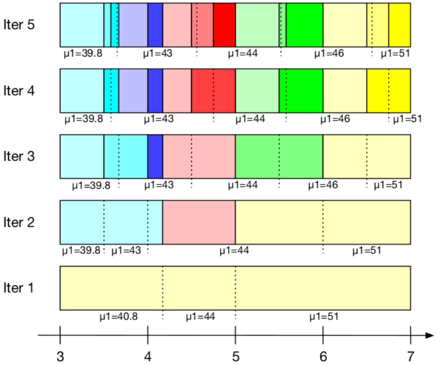
Numerical Results - LandS



Iter	x_1	x_2	x_3	x_4
1	0.833	3.000	4.167	4.000
2	2.500	3.000	3.500	3.000
3	1.833	4.000	3.667	2.500
4	2.000	4.167	3.583	2.250
5	1.917	4.083	3.625	2.375
6	1.875	4.042	3.646	2.438

Iter	LB	UB	Gap
1	378.667	382.711	1.0567%
2	380.122	381.100	0.2567%
3	380.601	380.844	0.0640%
4	380.842	380.893	0.0007%
5	380.843	380.856	0.0004%
6	380.844	380.847	0.0002%

Results given by GAPM for LandS problem[3]

[3] illustration from Ramirez-Pico and Moreno

Numerical Results - ProdMix

k	x_k	z_L^k	z_U^k	Gap	$\lvert\mathcal{P}_k^{\max}\rvert$
1	$(1333.33, 66.67)$	-18666.67	-16939.71	9.3%	4
2	$(1441.41, 59.57)$	-17873.01	-17383.73	2.7%	9
3	$(1399.05, 57.91)$	-17789.88	-17659.19	0.74%	16
4	$(1379.98, 56.64)$	-17744.67	-17708.00	0.20%	25
5	$(1371.36, 55.71)$	-17718.96	-17709.05	0.056%	36
6	$(1375.55, 56.21)$	-17713.74	-17711.37	0.013%	49

Table: Results for problem Prod-Mix

To compare our approach with SAA, we solved the same problem 100 times, each with 10 000 scenarios randomly drawn, yielding a 95% confidence interval centered in -17711, with radius 2.2.

Conclusion and applications

- *Uniform and universal* exact quantization for an MSLP
  - ➥ New complexity results.

  Unfortunately this quantization might be very large.

- *Local* exact quantization for $\boldsymbol{c}$
  - ➥ Higher order simplex algorithm on the chamber complex for 2SLP.

- *Local* exact quantization for $\boldsymbol{B}$ and $\boldsymbol{b}$.
  - ➥ Adaptive Partition-based Methods (APM) for general distribution: solves small 2SLP with high precision.

- Extension of Stochastic Dual Dynamic Programming algorithms for non finitely supported distribution.

- Links with fundamental polyhedral geometry, regular subdivisions and fiber polytope.

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Thank you for listening ! Any question ?

M. Forcier, S. Gaubert, V. Leclère
Exact quantization of multistage stochastic linear problems.
arXiv preprint arXiv:2107.09566 (2021).

M. Forcier, V. Leclère
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Operation Research Letters, to appear (2022).

M. Forcier, V. Leclère
Convergence of Trajectory Following Dynamic Programming algorithms for multistage stochastic problems without finite support assumptions
HAL Id : hal-03683697 (2022).