Multistage Stochastic Linear Problem and Polyhedral Geometry

Maël Forcier, Stéphane Gaubert, Vincent Leclère

July 2nd, 2021

mael.forcier@enpc.fr

Multistage stochastic linear programming (MSLP)

$$
\begin{aligned}
\min_{(\mathbf{x}_t)_{t\in[T]}} \quad & \mathbb{E}\Big[\sum_{t=1}^{T} \mathbf{c}_t^\top \mathbf{x}_t\Big] & \\
\text{s.t.} \quad & \mathbf{A}_t\mathbf{x}_t + \mathbf{B}_t\mathbf{x}_{t-1} \leqslant \mathbf{b}_t & \forall t \in [T] \\
& \mathbf{x}_t \text{ random variable in } \mathbb{R}^{n_t} & \forall t \in [T] \\
& \sigma(\mathbf{x}_t) \subset \sigma(\mathbf{c}_k, \mathbf{A}_k, \mathbf{B}_k, \mathbf{b}_k)_{k\leqslant t} & \forall t \in [T] \\
& \mathbf{x}_0 \equiv x_0 \text{ given} &
\end{aligned}
$$

where $\mathbf{c}_t \in \mathbb{R}^{n_t}$, $\mathbf{A}_t \in \mathbb{R}^{q_t \times n_{t-1}}$,$\mathbf{B}_t \in \mathbb{R}^{q_t \times n_t}$ and $\mathbf{b}_t \in \mathbb{R}^{q_t}$ are given random variables.

$(\mathbf{c}_t, \mathbf{A}_t, \mathbf{B}_t, \mathbf{b}_t)_{t\in[T]}$ is an independent sequence.

We set $V_{T+1} \equiv 0$ and:

$$
V_t(x_{t-1}) := \mathbb{E}\begin{bmatrix} \min\limits_{x_t\in\mathbb{R}^{n_t}} \mathbf{c}_t^\top x_t + V_{t+1}(x_t) \\ \text{s.t. } \mathbf{A}_t x_t + \mathbf{B}_t x_{t-1} \leqslant \mathbf{b}_t \end{bmatrix}
$$

Multistage stochastic linear programming (MSLP)

$$
\begin{aligned}
\min_{(\mathbf{x}_t)_{t\in[T]}} \quad & \mathbb{E}\Big[\sum_{t=1}^{T} \mathbf{c}_t^\top \mathbf{x}_t\Big] & \\
\text{s.t.} \quad & \mathbf{A}_t\mathbf{x}_t + \mathbf{B}_t\mathbf{x}_{t-1} \leqslant \mathbf{b}_t & \forall t \in [T] \\
& \mathbf{x}_t \text{ random variable in } \mathbb{R}^{n_t} & \forall t \in [T] \\
& \sigma(\mathbf{x}_t) \subset \sigma(\mathbf{c}_k, \mathbf{A}_k, \mathbf{B}_k, \mathbf{b}_k)_{k\leqslant t} & \forall t \in [T] \\
& \mathbf{x}_0 \equiv x_0 \text{ given} &
\end{aligned}
$$

where $\mathbf{c}_t \in \mathbb{R}^{n_t}$, $\mathbf{A}_t \in \mathbb{R}^{q_t\times n_{t-1}}$,$\mathbf{B}_t \in \mathbb{R}^{q_t\times n_t}$ and $\mathbf{b}_t \in \mathbb{R}^{q_t}$ are given random variables.
$(\mathbf{c}_t, \mathbf{A}_t, \mathbf{B}_t, \mathbf{b}_t)_{t\in[T]}$ is an independent sequence.

We set $V_{T+1} \equiv 0$ and:

$$
V_t(x_{t-1}) := \mathbb{E}\left[\begin{aligned} \min_{x_t\in\mathbb{R}^{n_t}} \; & \mathbf{c}_t^\top x_t + V_{t+1}(x_t) \\ \text{s.t.} \; & \mathbf{A}_t x_t + \mathbf{B}_t x_{t-1} \leqslant \mathbf{b}_t \end{aligned}\right]
$$

Quantization of a MSLP

The random variable $(\mathbf{c}_t, \mathbf{A}_t, \mathbf{B}_t, \mathbf{b}_t)_{t\in[T]}$ are often replaced by a discrete distribution on a finite number of scenarios

$$V_t(x_{t-1}) \simeq \widetilde{V}_t(x_{t-1}) = \sum_{k=1}^{K} p_k \begin{array}{rl} \min\limits_{x_t\in\mathbb{R}^{n_t}} & c_{t,k}^\top x_t + V_{t+1}(x_t) \\ \text{s.t.} & A_{t,k}x_t + B_{t,k}x_{t-1} \leqslant b_{t,k} \end{array}$$

Scenario drawn by Monte Carlo : Sample Average Approximation

Definition

We say that an MSLP admits an *exact quantization* if there exists a finitely supported $(\check{\mathbf{c}}_t, \check{\mathbf{A}}_t, \check{\mathbf{B}}_t, \check{\mathbf{b}}_t)_{t\in[T]}$ that yields the same expected cost-to-go functions, $(V_t)_{t\in[T]}$. In particular the MSLP is equivalent to a problem on a finite scenario tree.

Contents

Contents

Reformulation of $V(x)$ highlighting the role of the fiber P_x

For a given x,

$$V(x) := \mathbb{E}\left[\begin{array}{ll} \min\limits_{y \in \mathbb{R}^m} & \mathbf{c}^\top y \\ \text{s.t.} & Ax + By \leqslant b \end{array}\right]$$

$$V(x) = \mathbb{E}\Big[\min_{y \in P_x} \mathbf{c}^\top y\Big] \quad \text{where} \quad P_x := \{y \in \mathbb{R}^m \mid Ax + By \leqslant b\}$$

Illustrative running example:

$$P_x := \{y \in \mathbb{R}^m \mid \|y\|_1 \leqslant 1, \quad y_1 \leqslant x, \quad y_2 \leqslant x\}$$

P_x for $x = 0.8$

P_x for $x = 0.3$

Reformulation of $V(x)$ highlighting the role of the fiber P_x

For a given x,

$$V(x) := \mathbb{E}\begin{bmatrix} \min\limits_{y \in \mathbb{R}^m} \ \mathbf{c}^\top y \\ \text{s.t.} \quad Ax + By \leqslant b \end{bmatrix}$$

$$V(x) = \mathbb{E}\Big[\min_{y \in P_x} \mathbf{c}^\top y\Big] \quad \text{where} \quad P_x := \{y \in \mathbb{R}^m \mid Ax + By \leqslant b\}$$

Illustrative running example:

$$P_x := \{y \in \mathbb{R}^m \mid \|y\|_1 \leqslant 1, \quad y_1 \leqslant x, \quad y_2 \leqslant x\}$$

P_x for $x = 0.8$

P_x for $x = 0.3$

Reformulation of $V(x)$ highlighting the role of the fiber P_x

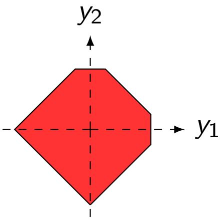
For a given x,

$$V(x) := \mathbb{E}\left[\begin{array}{ll}\min\limits_{y\in\mathbb{R}^m} & \mathbf{c}^\top y \\ \text{s.t.} & Ax + By \leqslant b\end{array}\right]$$

$$V(x) = \mathbb{E}\big[\min_{y\in P_x} \mathbf{c}^\top y\big] \quad \text{where} \quad P_x := \{y\in\mathbb{R}^m \mid Ax + By \leqslant b\}$$

Illustrative running example:

$$P_x := \{y \in \mathbb{R}^m \mid \|y\|_1 \leqslant 1, \quad y_1 \leqslant x, \quad y_2 \leqslant x\}$$



P_x for $x = 0.8$

P_x for $x = 0.3$

Reformulation of $V(x)$ highlighting the role of the fiber P_x

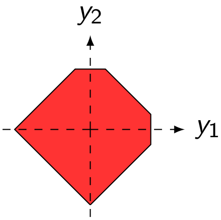
For a given x,

$$V(x) := \mathbb{E}\left[\begin{array}{ll}\min\limits_{y\in\mathbb{R}^m} & \mathbf{c}^\top y \\ \text{s.t.} & Ax + By \leqslant b\end{array}\right]$$

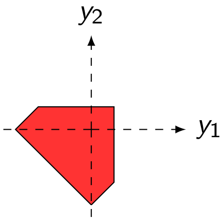
$$V(x) = \mathbb{E}\Big[\min_{y\in P_x} \mathbf{c}^\top y\Big] \quad \text{where} \quad P_x := \{y \in \mathbb{R}^m \mid Ax + By \leqslant b\}$$

Illustrative running example:

$$P_x := \{y \in \mathbb{R}^m \mid \|y\|_1 \leqslant 1, \quad y_1 \leqslant x, \quad y_2 \leqslant x\}$$



P_x for $x = 0.8$



P_x for $x = 0.3$

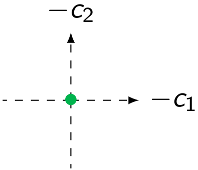
Normal fan $\mathcal{N}(P_x)$

Definition

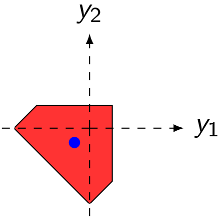
The normal fan of the fiber P_x *is*

$$\mathcal{N}(P_x) := \{N_{P_x}(y) \mid y \in P_x\}$$

with $N_{P_x}(y) = \{c \mid \forall y' \in P_x,\ c^\top(y' - y) \leqslant 0\}$ *the normal cone of* P_x *on* y.



$N_{P_x}(y)$ for $x = 0.3$



P_x y and $N_{P_x}(y)$ for $x = 0.3$

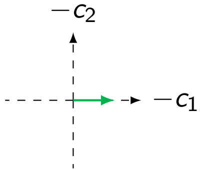
Normal fan $\mathcal{N}(P_x)$

Definition

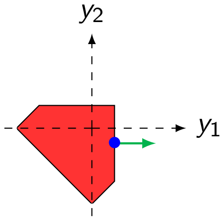
The normal fan of the fiber P_x is

$$\mathcal{N}(P_x) := \{N_{P_x}(y) \mid y \in P_x\}$$

with $N_{P_x}(y) = \{c \mid \forall y' \in P_x,\ c^\top (y' - y) \leqslant 0\}$ *the normal cone of* P_x *on* y.



$N_{P_x}(y)$ for $x = 0.3$



P_x y and $N_{P_x}(y)$ for $x = 0.3$

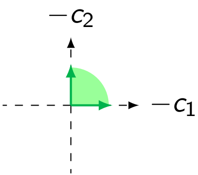
Normal fan $\mathcal{N}(P_x)$

Definition

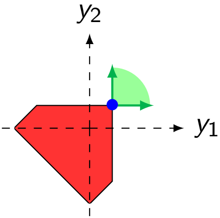
The normal fan of the fiber P_x is

$$\mathcal{N}(P_x) := \{N_{P_x}(y) \mid y \in P_x\}$$

with $N_{P_x}(y) = \{c \mid \forall y' \in P_x,\ c^\top(y' - y) \leqslant 0\}$ *the normal cone of* P_x *on* y.



$N_{P_x}(y)$ for $x = 0.3$



P_x y and $N_{P_x}(y)$ for $x = 0.3$

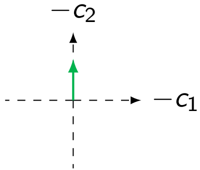
Normal fan $\mathcal{N}(P_x)$

Definition

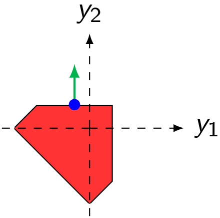
The normal fan of the fiber P_x is

$$\mathcal{N}(P_x) := \{N_{P_x}(y) \mid y \in P_x\}$$

with $N_{P_x}(y) = \{c \mid \forall y' \in P_x,\ c^\top(y' - y) \leqslant 0\}$ *the normal cone of P_x on y.*



$N_{P_x}(y)$ for $x = 0.3$



P_x y and $N_{P_x}(y)$ for $x = 0.3$

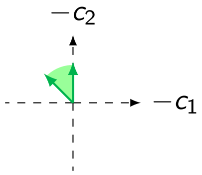
Normal fan $\mathcal{N}(P_x)$

Definition

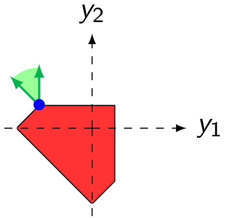
The normal fan of the fiber P_x *is*

$$\mathcal{N}(P_x) := \{N_{P_x}(y) \mid y \in P_x\}$$

with $N_{P_x}(y) = \{c \mid \forall y' \in P_x,\ c^\top(y' - y) \leqslant 0\}$ *the normal cone of* P_x *on* y.



$N_{P_x}(y)$ for $x = 0.3$



P_x y and $N_{P_x}(y)$ for $x = 0.3$

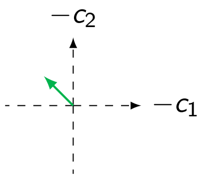
Normal fan $\mathcal{N}(P_x)$

Definition

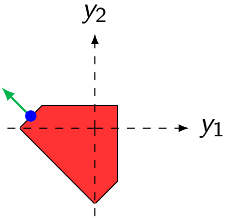
The normal fan of the fiber P_x *is*

$$\mathcal{N}(P_x) := \{N_{P_x}(y) \,|\, y \in P_x\}$$

with $N_{P_x}(y) = \{c \,|\, \forall y' \in P_x,\ c^\top(y' - y) \leqslant 0\}$ *the normal cone of* P_x *on* y.



$N_{P_x}(y)$ for $x = 0.3$



P_x y and $N_{P_x}(y)$ for $x = 0.3$

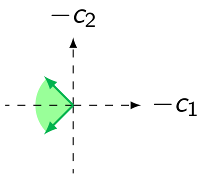
Normal fan $\mathcal{N}(P_x)$

Definition

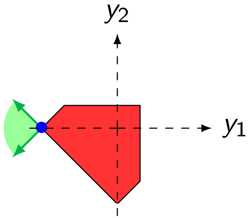
The normal fan of the fiber P_x *is*

$$\mathcal{N}(P_x) := \{N_{P_x}(y) \mid y \in P_x\}$$

with $N_{P_x}(y) = \{c \mid \forall y' \in P_x,\ c^\top(y' - y) \leqslant 0\}$ *the normal cone of* P_x *on* y.



$N_{P_x}(y)$ for $x = 0.3$



P_x y and $N_{P_x}(y)$ for $x = 0.3$

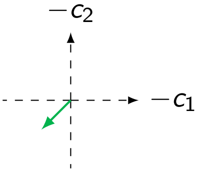
Normal fan $\mathcal{N}(P_x)$

Definition

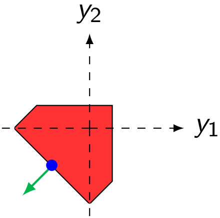
The normal fan of the fiber P_x is

$$\mathcal{N}(P_x) := \{N_{P_x}(y) \,|\, y \in P_x\}$$

with $N_{P_x}(y) = \{c \,|\, \forall y' \in P_x,\ c^\top (y' - y) \leqslant 0\}$ *the normal cone of P_x on y.*



$N_{P_x}(y)$ for $x = 0.3$



P_x y and $N_{P_x}(y)$ for $x = 0.3$

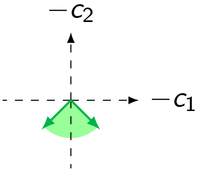
Normal fan $\mathcal{N}(P_x)$

Definition

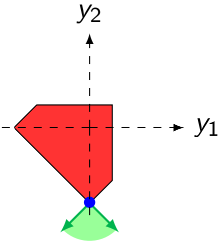
The normal fan of the fiber P_x is

$$\mathcal{N}(P_x) := \{N_{P_x}(y) \mid y \in P_x\}$$

with $N_{P_x}(y) = \{c \mid \forall y' \in P_x,\ c^\top (y' - y) \leqslant 0\}$ *the normal cone of P_x on y.*



$N_{P_x}(y)$ for $x = 0.3$



P_x y and $N_{P_x}(y)$ for $x = 0.3$

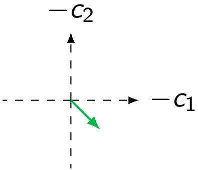
Normal fan $\mathcal{N}(P_x)$

Definition

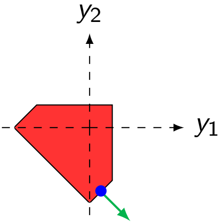
The normal fan of the fiber P_x is

$$\mathcal{N}(P_x) := \{N_{P_x}(y) \mid y \in P_x\}$$

with $N_{P_x}(y) = \{c \mid \forall y' \in P_x,\ c^\top(y' - y) \leqslant 0\}$ the normal cone of P_x on y.



$N_{P_x}(y)$ for $x = 0.3$



P_x y and $N_{P_x}(y)$ for $x = 0.3$

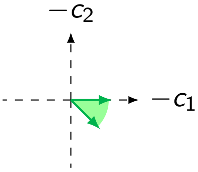
Normal fan $\mathcal{N}(P_x)$

Definition

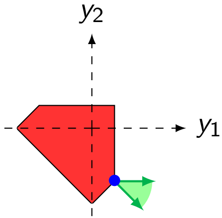
The normal fan of the fiber P_x *is*

$$\mathcal{N}(P_x) := \{N_{P_x}(y) \mid y \in P_x\}$$

with $N_{P_x}(y) = \{c \mid \forall y' \in P_x,\ c^\top(y' - y) \leqslant 0\}$ *the normal cone of* P_x *on* y.



$N_{P_x}(y)$ for $x = 0.3$



P_x y and $N_{P_x}(y)$ for $x = 0.3$

Normal fan $\mathcal{N}(P_x)$

Definition

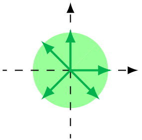
The normal fan of the fiber P_x is

$$\mathcal{N}(P_x) := \{N_{P_x}(y) \mid y \in P_x\}$$

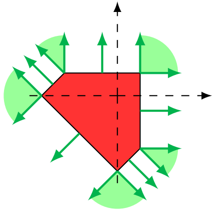
with $N_{P_x}(y) = \{c \mid \forall y' \in P_x,\ c^\top(y' - y) \leqslant 0\}$ *the normal cone of P_x on y.*

Proposition

If P_x is bounded, $\{\mathrm{ri}(N) \mid N \in \mathcal{N}(P_x)\}$ *is a partition of* $\mathbb{R}^m$.



$\mathcal{N}(P_x)$ for $x = 0.3$



P_x and $\mathcal{N}(P_x)$ for $x = 0.3$

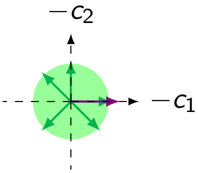
$\mathcal{N}(P_x)$: partition of cost coherent with the min

For a given x, we have

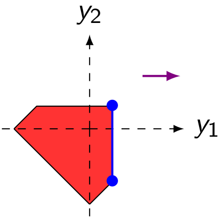
$$V(x) = \mathbb{E}\Big[\min_{y \in P_x} \mathbf{c}^\top y\Big]$$

For any $N \in \mathcal{N}(P_x)$ and $-c \to \arg\min_{y \in P_x} c^\top y$ is constant for all $-c \in \mathrm{ri}(N)$.

$\arg\min_{y \in P_x} c^\top y$ is a face of P_x.



Cost $-c$ and $\mathcal{N}(P_x)$ for $x = 0.3$



P_x for $x = 0.3$

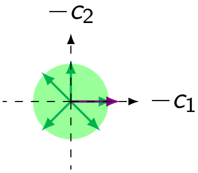
$\mathcal{N}(P_x)$: partition of cost coherent with the min

For a given x, we have

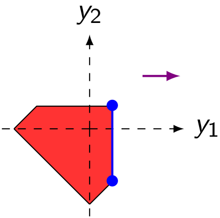
$$V(x) = \mathbb{E}\big[\min_{y \in P_x} \mathbf{c}^\top y\big]$$

For any $N \in \mathcal{N}(P_x)$ and $-c \to \arg\min_{y \in P_x} c^\top y$ is constant for all $-c \in \mathrm{ri}(N)$.

$\arg\min_{y \in P_x} c^\top y$ is a face of P_x.



Cost $-c$ and $\mathcal{N}(P_x)$ for $x = 0.3$



P_x for $x = 0.3$

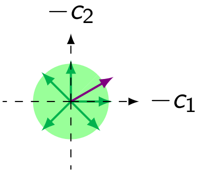
$\mathcal{N}(P_x)$: partition of cost coherent with the min

For a given x, we have

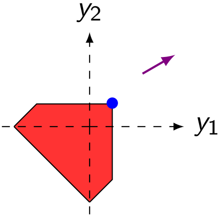
$$V(x) = \mathbb{E}\big[\min_{y \in P_x} \mathbf{c}^\top y\big]$$

For any $N \in \mathcal{N}(P_x)$ and $-c \to \arg\min_{y \in P_x} c^\top y$ is constant for all $-c \in \mathrm{ri}(N)$.

$\arg\min_{y \in P_x} c^\top y$ is a face of P_x.



Cost $-c$ and $\mathcal{N}(P_x)$ for $x = 0.3$



P_x for $x = 0.3$

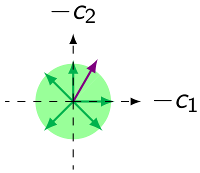
$\mathcal{N}(P_x)$: partition of cost coherent with the min

For a given x, we have

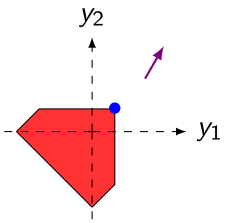
$$V(x) = \mathbb{E}\big[\min_{y \in P_x} \mathbf{c}^\top y\big]$$

For any $N \in \mathcal{N}(P_x)$ and $-c \to \arg\min_{y \in P_x} c^\top y$ is constant for all $-c \in \mathrm{ri}(N)$.

$\arg\min_{y \in P_x} c^\top y$ is a face of P_x.



Cost $-c$ and $\mathcal{N}(P_x)$ for $x = 0.3$



P_x for $x = 0.3$

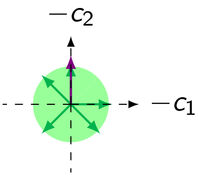
$\mathcal{N}(P_x)$: partition of cost coherent with the min

For a given x, we have

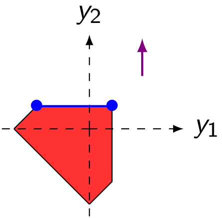
$$V(x) = \mathbb{E}\Big[\min_{y \in P_x} \mathbf{c}^\top y\Big]$$

For any $N \in \mathcal{N}(P_x)$ and $-c \to \arg\min_{y \in P_x} c^\top y$ is constant for all $-c \in \mathrm{ri}(N)$.

$\arg\min_{y \in P_x} c^\top y$ is a face of P_x.



Cost $-c$ and $\mathcal{N}(P_x)$ for $x = 0.3$



P_x for $x = 0.3$

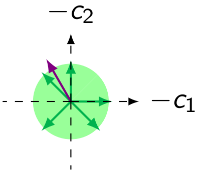
$\mathcal{N}(P_x)$: partition of cost coherent with the min

For a given x, we have

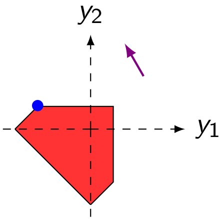
$$V(x) = \mathbb{E}\big[\min_{y \in P_x} \mathbf{c}^\top y\big]$$

For any $N \in \mathcal{N}(P_x)$ and $-c \to \arg\min_{y \in P_x} c^\top y$ is constant for all $-c \in \mathrm{ri}(N)$.

$\arg\min_{y \in P_x} c^\top y$ is a face of P_x.



Cost $-c$ and $\mathcal{N}(P_x)$ for $x = 0.3$



P_x for $x = 0.3$

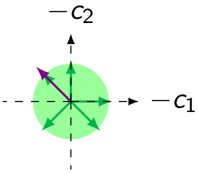
$\mathcal{N}(P_x)$: partition of cost coherent with the min

For a given x, we have

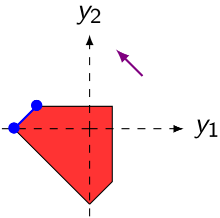
$$V(x) = \mathbb{E}\Big[\min_{y \in P_x} \mathbf{c}^\top y\Big]$$

For any $N \in \mathcal{N}(P_x)$ and $-c \to \arg\min_{y \in P_x} c^\top y$ is constant for all $-c \in \mathrm{ri}(N)$.

$\arg\min_{y \in P_x} c^\top y$ is a face of P_x.



Cost $-c$ and $\mathcal{N}(P_x)$ for $x = 0.3$



P_x for $x = 0.3$

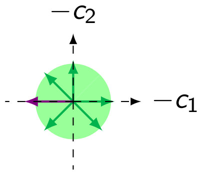
$\mathcal{N}(P_x)$: partition of cost coherent with the min

For a given x, we have

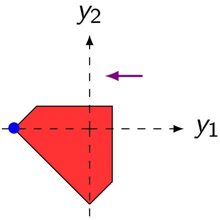
$$V(x) = \mathbb{E}\Big[\min_{y \in P_x} \mathbf{c}^\top y\Big]$$

For any $N \in \mathcal{N}(P_x)$ and $-c \to \arg\min_{y \in P_x} c^\top y$ is constant for all $-c \in \mathrm{ri}(N)$.

$\arg\min_{y \in P_x} c^\top y$ is a face of P_x.



Cost $-c$ and $\mathcal{N}(P_x)$ for $x = 0.3$



P_x for $x = 0.3$

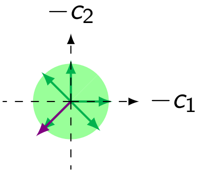
$\mathcal{N}(P_x)$: partition of cost coherent with the min

For a given x, we have

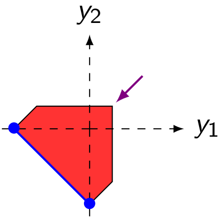
$$V(x) = \mathbb{E}\big[\min_{y \in P_x} \mathbf{c}^\top y\big]$$

For any $N \in \mathcal{N}(P_x)$ and $-c \to \arg\min_{y \in P_x} c^\top y$ is constant for all $-c \in \mathrm{ri}(N)$.

$\arg\min_{y \in P_x} c^\top y$ is a face of P_x.



Cost $-c$ and $\mathcal{N}(P_x)$ for $x = 0.3$



P_x for $x = 0.3$

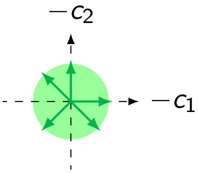
$\mathcal{N}(P_x)$: partition of cost coherent with the min

For a given x, we have

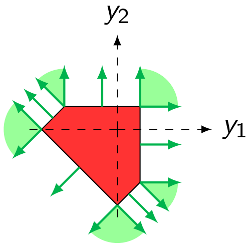
$$V(x) = \mathbb{E}\big[\min_{y \in P_x} \mathbf{c}^\top y\big]$$

For any $N \in \mathcal{N}(P_x)$ and $-c \to \arg\min_{y \in P_x} c^\top y$ is constant for all $-c \in \mathrm{ri}(N)$.

$\arg\min_{y \in P_x} c^\top y$ is a face of P_x.



$\mathcal{N}(P_x)$ for $x = 0.3$



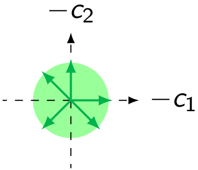
P_x for $x = 0.3$

Reduction to a finite sum

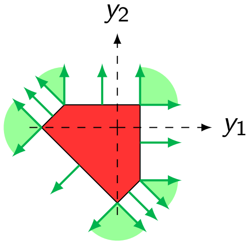
For a fixed x,

$$V(x) = \mathbb{E}\big[\min_{y \in P_x} \mathbf{c}^\top y\big] = \sum_{N \in \mathcal{N}(P_x)} \mathbb{E}\big[\mathbf{c}^\top \mathbb{1}_{\mathbf{c} \in -\operatorname{ri} N}\big] y_N(x)$$

where $y_N(x) \in \arg\min_{y \in P_x} c^\top y$ for any $c \in \operatorname{ri}(N)$.



$\mathcal{N}(P_x)$ for $x = 0.3$

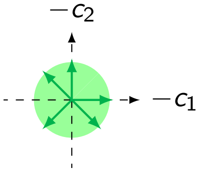


P_x and $\mathcal{N}(P_x)$ for $x = 0.3$

General cost $\mathbf{c}$ is equivalent to discrete cost $\check{\mathbf{c}}$ for given x

For a fixed x,

$$
\begin{aligned}
V(x) &= \mathbb{E}\Big[\min_{y \in P_x} \mathbf{c}^\top y\Big] \\
&= \sum_{N \in \mathcal{N}(P_x)} \mathbb{E}\big[\mathbf{c}^\top \mathbb{1}_{\mathbf{c} \in -\operatorname{ri} N}\big] y_N(x)
\end{aligned}
$$



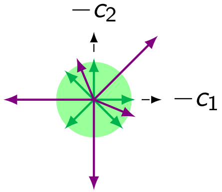
$\mathcal{N}(P_x)$ for $x = 0.3$

We draw a continuous cost $\mathbf{c}$.

General cost $\mathbf{c}$ is equivalent to discrete cost $\check{\mathbf{c}}$ for given x

For a fixed x,

$$
\begin{aligned}
V(x) &= \mathbb{E}\big[\min_{y \in P_x} \mathbf{c}^\top y\big] \\
&= \sum_{N \in \mathcal{N}(P_x)} \mathbb{E}\big[\mathbf{c}^\top \mathbb{1}_{\mathbf{c} \in -\operatorname{ri} N}\big] y_N(x) \\
&= \sum_{N \in \mathcal{N}(P_x)} p_N \check{c}_N^\top y_N(x)
\end{aligned}
$$



$\mathcal{N}(P_x)$ and $p_N \check{c}_N$ for $x = 0.3$

For $N \in \mathcal{N}(P_x)$,

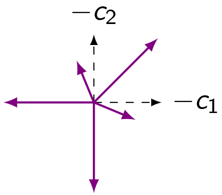
$$
\begin{aligned}
p_N &:= \mathbb{P}\big[\mathbf{c} \in -\operatorname{ri} N\big] \\
\check{c}_N &:= \mathbb{E}\big[\mathbf{c} | \mathbf{c} \in -\operatorname{ri} N\big]
\end{aligned}
$$

Instead of drawing a general $\mathbf{c}$, we draw a discrete cost $\check{\mathbf{c}}$ indexed by the finite collection $\mathcal{N}(P_x)$.

General cost $\mathbf{c}$ is equivalent to discrete cost $\check{\mathbf{c}}$ for given x

For a fixed x,

$$\begin{aligned} V(x) &= \mathbb{E}\Big[\min_{y \in P_x} \mathbf{c}^\top y\Big] \\ &= \sum_{N \in \mathcal{N}(P_x)} \mathbb{E}\big[\mathbf{c}^\top \mathbb{1}_{\mathbf{c} \in -\operatorname{ri} N}\big] y_N(x) \\ &= \sum_{N \in \mathcal{N}(P_x)} p_N \check{c}_N{}^\top y_N(x) \\ &= \sum_{N \in \mathcal{N}(P_x)} p_N \min_{y \in P_x} \check{c}_N{}^\top y \end{aligned}$$



$p_N \check{c}_N$ for $x = 0.3$

For $N \in \mathcal{N}(P_x)$,

$$\begin{aligned} p_N &:= \mathbb{P}\big[\mathbf{c} \in -\operatorname{ri} N\big] \\ \check{c}_N &:= \mathbb{E}\big[\mathbf{c} | \mathbf{c} \in -\operatorname{ri} N\big] \end{aligned}$$

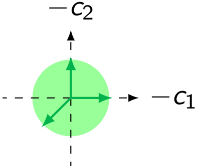
Instead of drawing a general $\mathbf{c}$, we draw a discrete cost $\check{\mathbf{c}}$ indexed by the finite collection $\mathcal{N}(P_x)$.

Contents

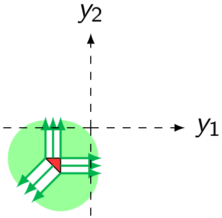
$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x, y) \mid Ax + By \leqslant b\} \quad \text{and} \quad P_x := \{y \mid Ax + By \leqslant b\}$$

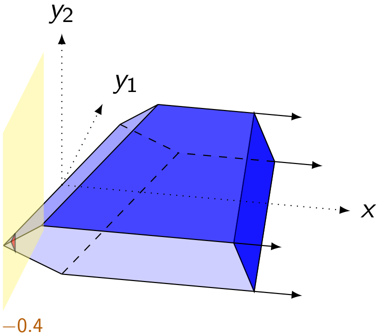
$$x = -0.4$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

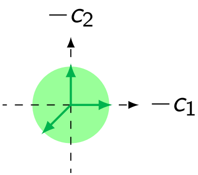


P and P_x

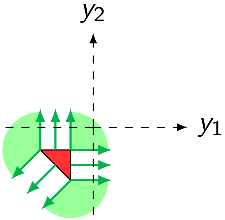
$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x, y) \mid Ax + By \leqslant b\} \quad \text{and} \quad P_x := \{y \mid Ax + By \leqslant b\}$$

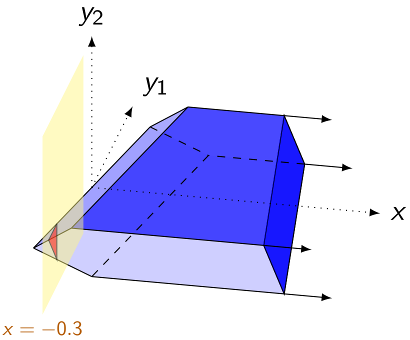
$$x = -0.3$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

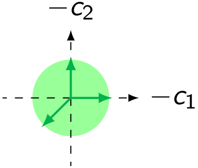


P and P_x

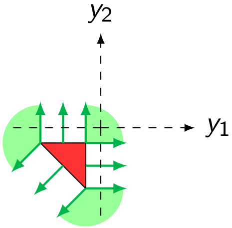
$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x, y) \mid Ax + By \leqslant b\} \quad \text{and} \quad P_x := \{y \mid Ax + By \leqslant b\}$$

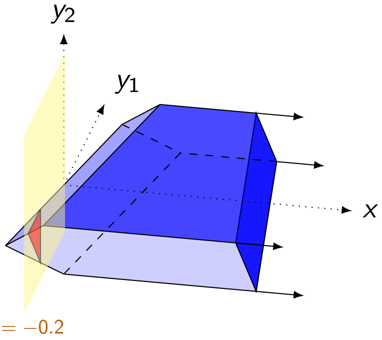
$$x = -0.2$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

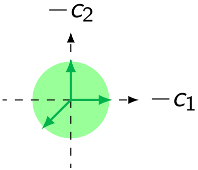


P and P_x

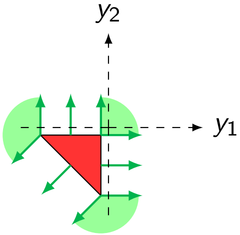
$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x,y) \mid Ax + By \leqslant b\} \quad \text{and} \quad P_x := \{y \mid Ax + By \leqslant b\}$$

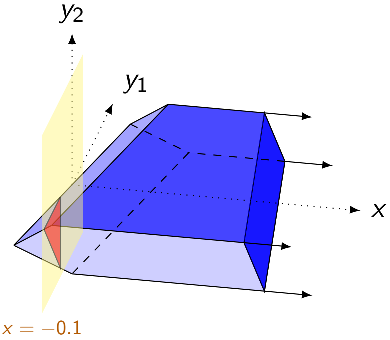
$$x = -0.1$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

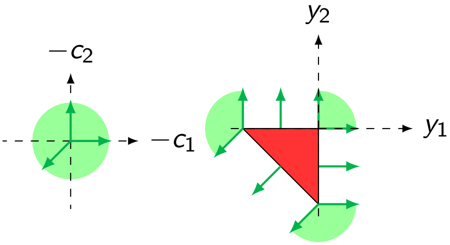


P and P_x

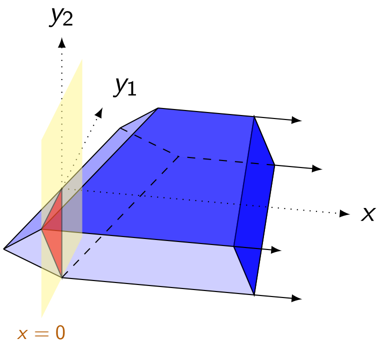
$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x, y) \mid Ax + By \leqslant b\} \quad \text{and} \quad P_x := \{y \mid Ax + By \leqslant b\}$$

$x = 0$



$\mathcal{N}(P_x)$ $\quad$ P_x and $\mathcal{N}(P_x)$

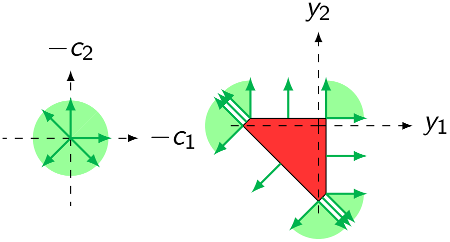


P and P_x

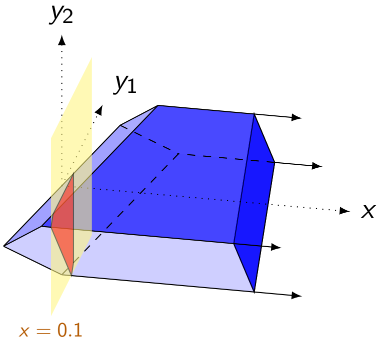
$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x, y) \mid Ax + By \leqslant b\} \quad \text{and} \quad P_x := \{y \mid Ax + By \leqslant b\}$$

$x = 0.1$



$\mathcal{N}(P_x)$ $\quad$ P_x and $\mathcal{N}(P_x)$

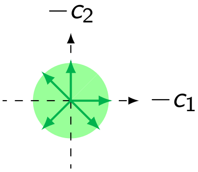


P and P_x

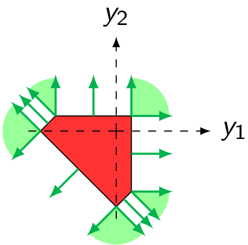
$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x,y) \mid Ax + By \leqslant b\} \quad \text{and} \quad P_x := \{y \mid Ax + By \leqslant b\}$$

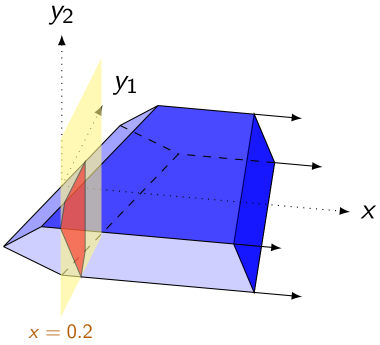
$$x = 0.2$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

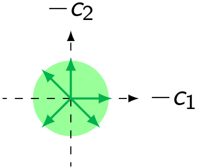


P and P_x

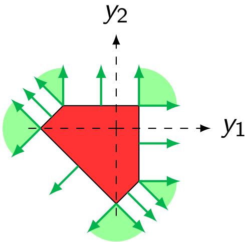
$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x,y) \mid Ax + By \leqslant b\} \quad \text{and} \quad P_x := \{y \mid Ax + By \leqslant b\}$$

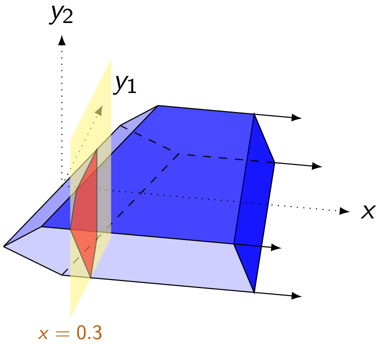
$x = 0.3$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

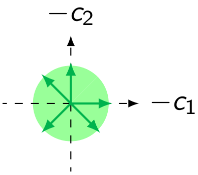


P and P_x

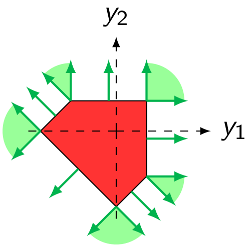
$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x,y) \mid Ax + By \leqslant b\} \quad \text{and} \quad P_x := \{y \mid Ax + By \leqslant b\}$$

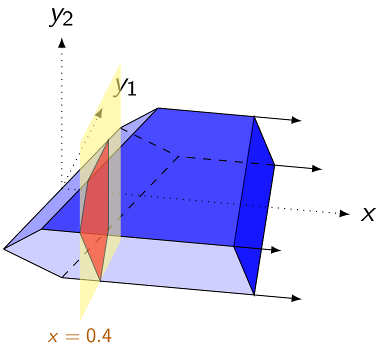
$x = 0.4$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

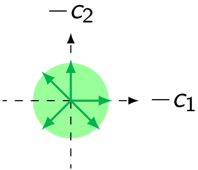


P and P_x

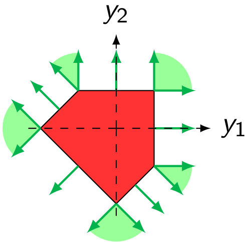
$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x,y) \mid Ax + By \leqslant b\} \quad \text{and} \quad P_x := \{y \mid Ax + By \leqslant b\}$$

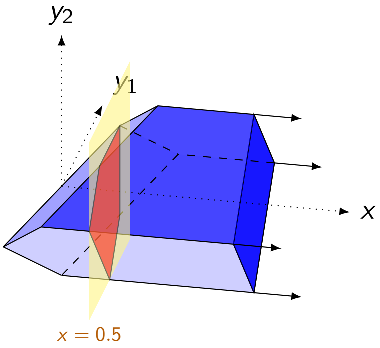
$$x = 0.5$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

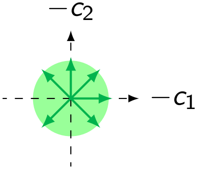


P and P_x

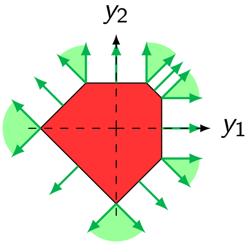
$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x, y) \mid Ax + By \leqslant b\} \quad \text{and} \quad P_x := \{y \mid Ax + By \leqslant b\}$$

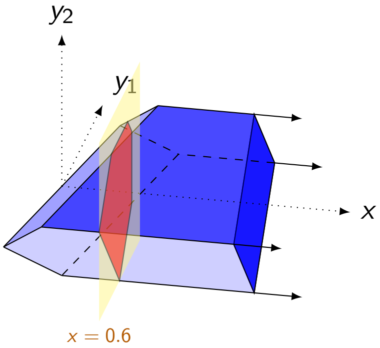
$$x = 0.6$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

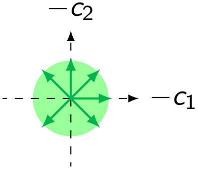


P and P_x

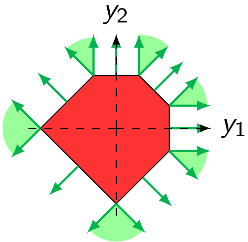
$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x, y) \mid Ax + By \leqslant b\} \quad \text{and} \quad P_x := \{y \mid Ax + By \leqslant b\}$$

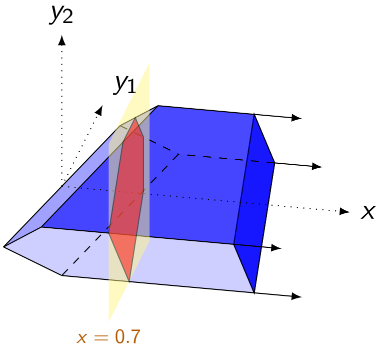
$x = 0.7$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

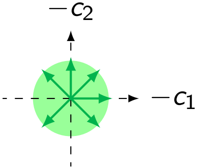


P and P_x

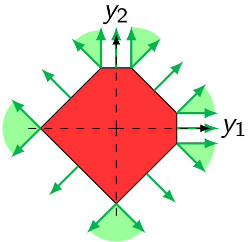
$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x, y) \mid Ax + By \leqslant b\} \quad \text{and} \quad P_x := \{y \mid Ax + By \leqslant b\}$$

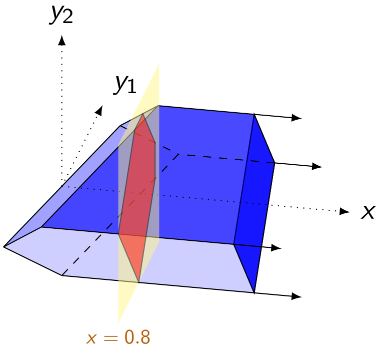
$x = 0.8$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

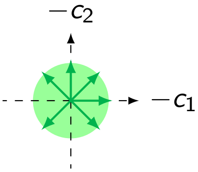


P and P_x

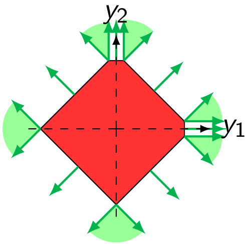
$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x,y) \mid Ax + By \leqslant b\} \quad \text{and} \quad P_x := \{y \mid Ax + By \leqslant b\}$$

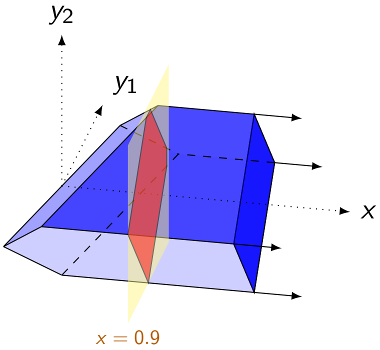
$x = 0.9$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

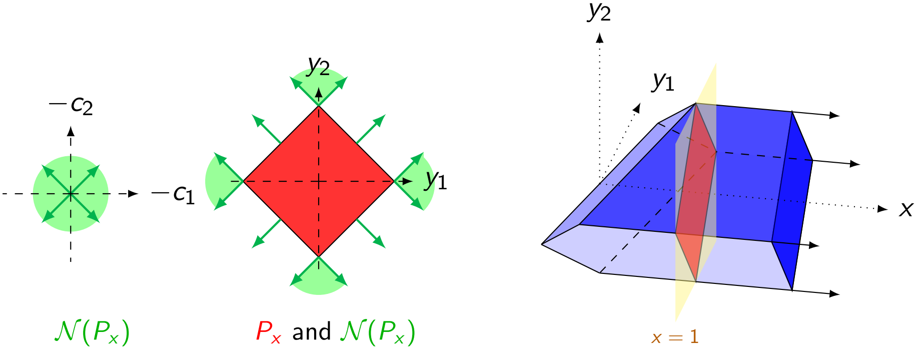


P and P_x

$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x, y) \mid Ax + By \leqslant b\} \quad \text{and} \quad P_x := \{y \mid Ax + By \leqslant b\}$$

$x = 1$

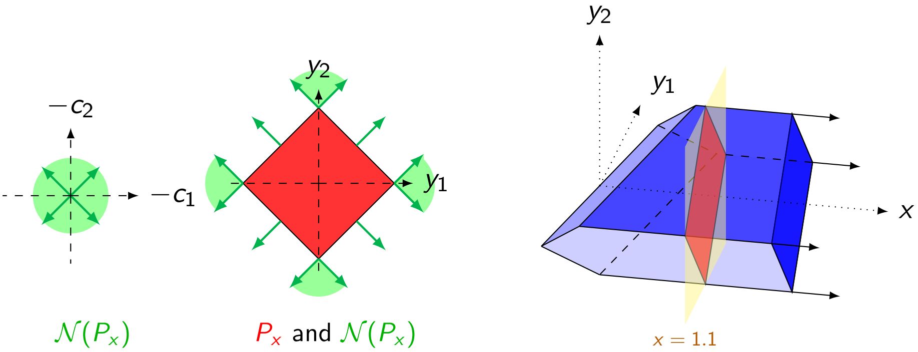


$\mathcal{N}(P_x)$ P_x and $\mathcal{N}(P_x)$ P and P_x

$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x,y) \mid Ax + By \leqslant b\} \quad \text{and} \quad P_x := \{y \mid Ax + By \leqslant b\}$$

$x = 1.1$

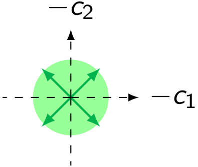


$\mathcal{N}(P_x)$ $\quad$ P_x and $\mathcal{N}(P_x)$ $\quad$ P and P_x

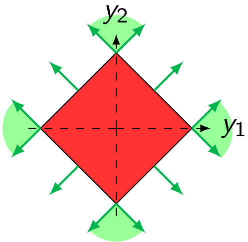
$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x,y) \mid Ax + By \leqslant b\} \quad \text{and} \quad P_x := \{y \mid Ax + By \leqslant b\}$$

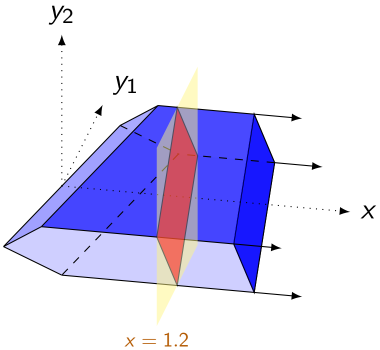
$x = 1.2$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

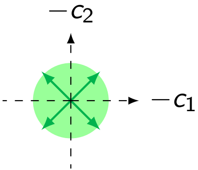


P and P_x

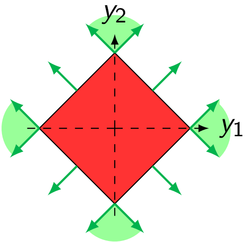
$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x,y) \mid Ax + By \leqslant b\} \quad \text{and} \quad P_x := \{y \mid Ax + By \leqslant b\}$$

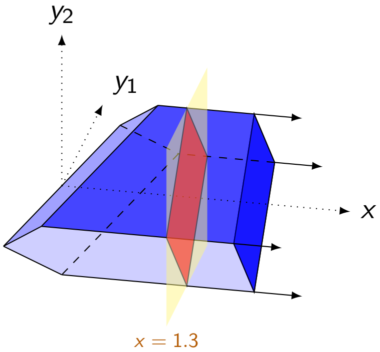
$$x = 1.3$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

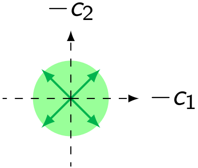


P and P_x

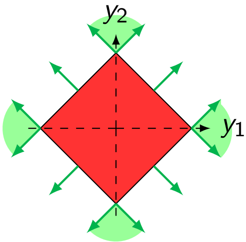
$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x, y) \mid Ax + By \leqslant b\} \quad \text{and} \quad P_x := \{y \mid Ax + By \leqslant b\}$$

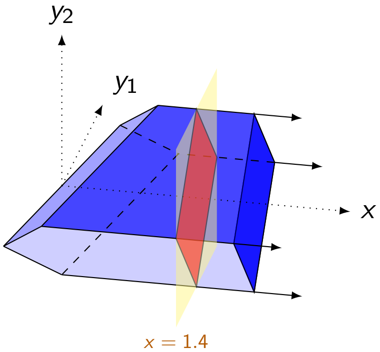
$x = 1.4$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

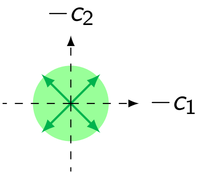


P and P_x

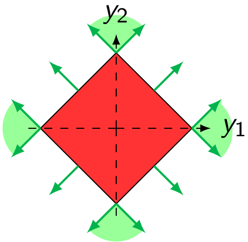
$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x,y) \mid Ax + By \leqslant b\} \quad \text{and} \quad P_x := \{y \mid Ax + By \leqslant b\}$$

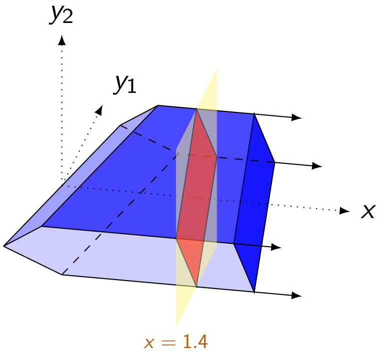
$x = 1.4$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

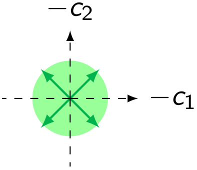


P and P_x

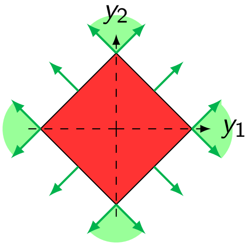
$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x, y) \mid Ax + By \leqslant b\} \quad \text{and} \quad P_x := \{y \mid Ax + By \leqslant b\}$$

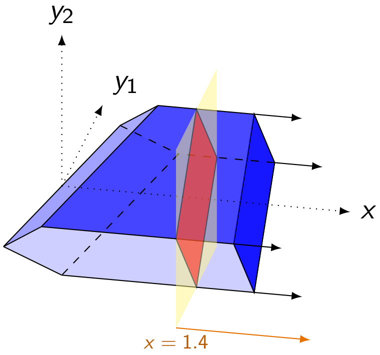
$x = 1.4$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

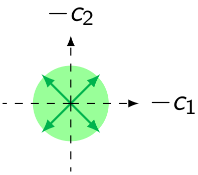


P and P_x

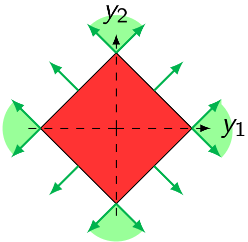
$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x,y) \mid Ax + By \leqslant b\} \quad \text{and} \quad P_x := \{y \mid Ax + By \leqslant b\}$$

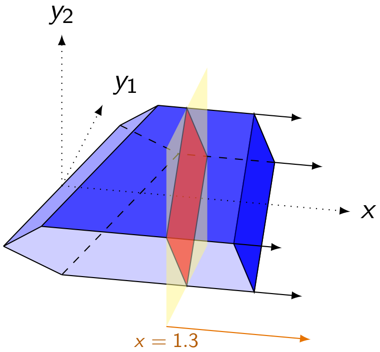
$x = 1.3$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

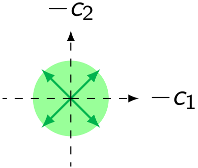


P and P_x

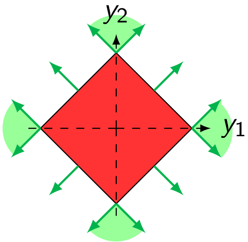
$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x, y) \mid Ax + By \leqslant b\} \quad \text{and} \quad P_x := \{y \mid Ax + By \leqslant b\}$$

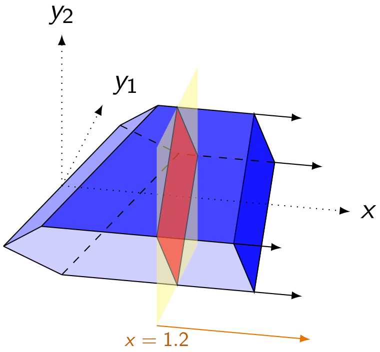
$x = 1.2$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

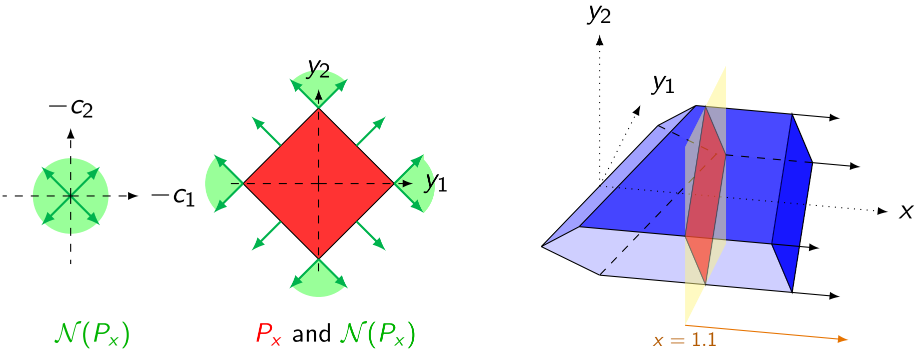


P and P_x

$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x,y) \mid Ax + By \leqslant b\} \quad \text{and} \quad P_x := \{y \mid Ax + By \leqslant b\}$$

$$x = 1.1$$

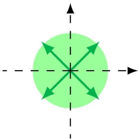


$\mathcal{N}(P_x)$ $\quad$ P_x and $\mathcal{N}(P_x)$ $\quad$ P and P_x

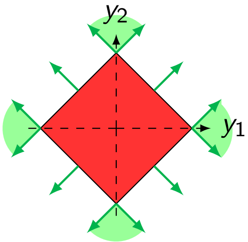
$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x, y) \mid Ax + By \leqslant b\} \quad \text{and} \quad P_x := \{y \mid Ax + By \leqslant b\}$$

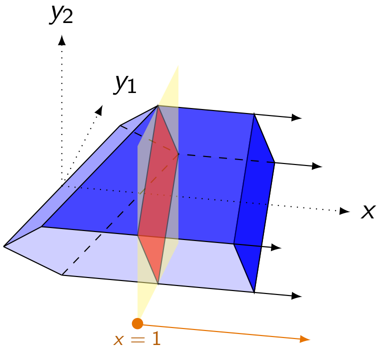
$x = 1$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

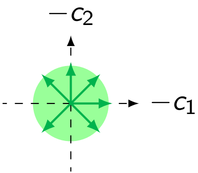


P and P_x

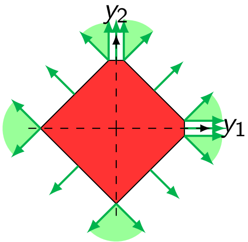
$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x, y) \mid Ax + By \leqslant b\} \quad \text{and} \quad P_x := \{y \mid Ax + By \leqslant b\}$$

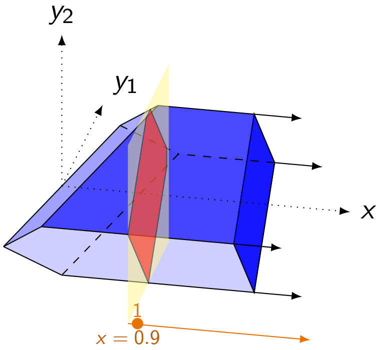
$$x = 0.9$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

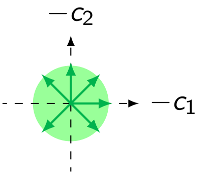


P and P_x

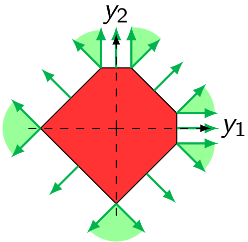
$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x,y) \mid Ax + By \leqslant b\} \quad \text{and} \quad P_x := \{y \mid Ax + By \leqslant b\}$$

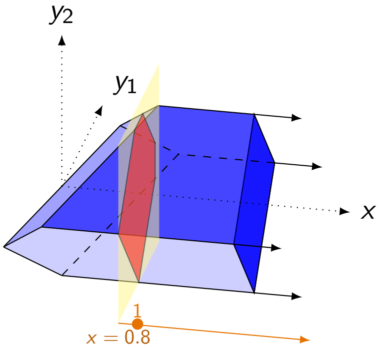
$x = 0.8$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

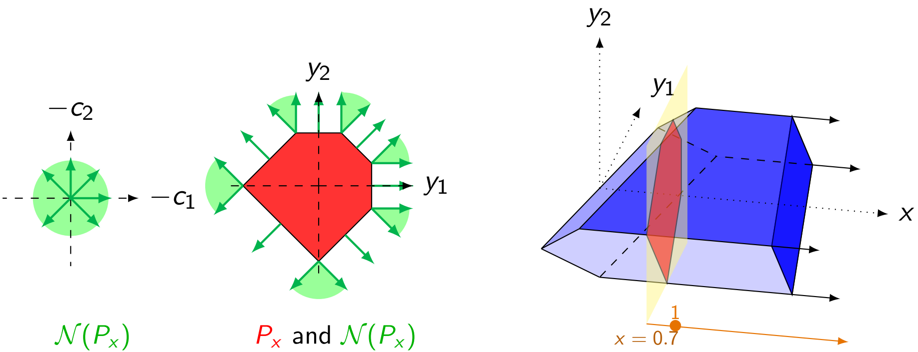


P and P_x

$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x, y) \mid Ax + By \leqslant b\} \quad \text{and} \quad P_x := \{y \mid Ax + By \leqslant b\}$$

$$x = 0.7$$

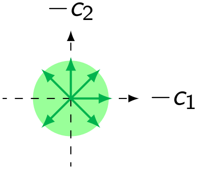


$\mathcal{N}(P_x)$ P_x and $\mathcal{N}(P_x)$ P and P_x

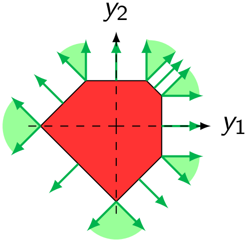
$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x, y) \mid Ax + By \leqslant b\} \quad \text{and} \quad P_x := \{y \mid Ax + By \leqslant b\}$$

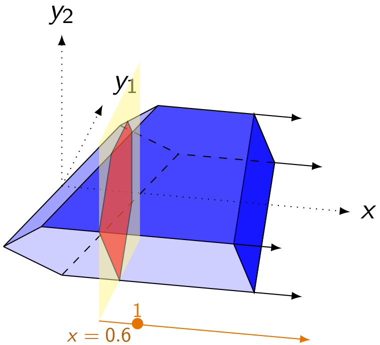
$x = 0.6$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

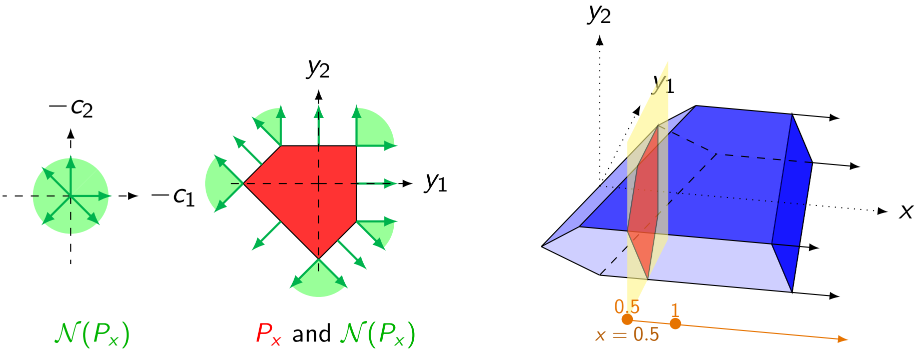


P and P_x

$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x,y) \mid Ax + By \leqslant b\} \quad \text{and} \quad P_x := \{y \mid Ax + By \leqslant b\}$$

$x = 0.5$

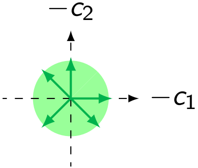


$\mathcal{N}(P_x)$ $\qquad$ P_x and $\mathcal{N}(P_x)$ $\qquad$ P and P_x

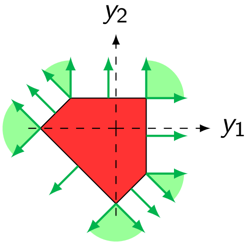
$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x,y) \mid Ax + By \leqslant b\} \quad \text{and} \quad P_x := \{y \mid Ax + By \leqslant b\}$$

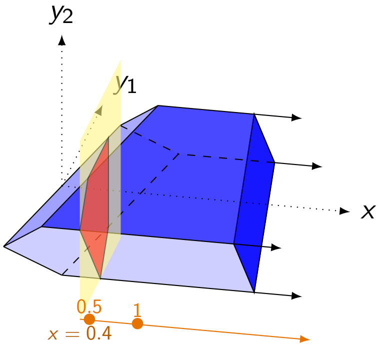
$$x = 0.4$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

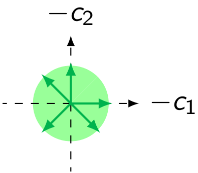


P and P_x

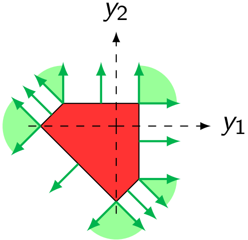
$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x, y) \mid Ax + By \leqslant b\} \quad \text{and} \quad P_x := \{y \mid Ax + By \leqslant b\}$$

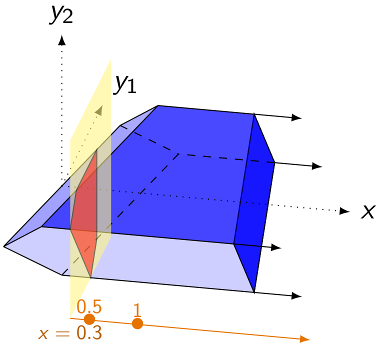
$x = 0.3$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

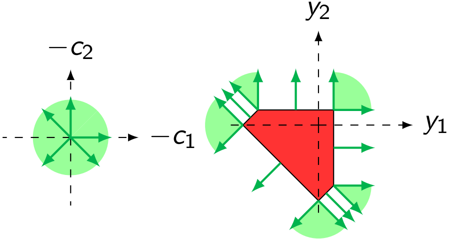


P and P_x

$\mathcal{N}(P_x)$ is piecewise constant with x.

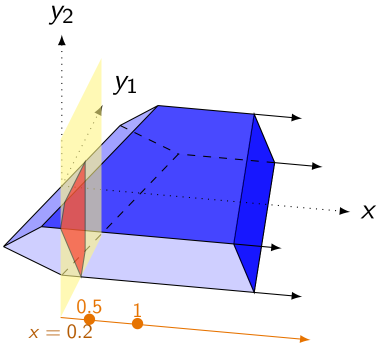
$$P := \{(x, y) \mid Ax + By \leqslant b\} \quad \text{and} \quad P_x := \{y \mid Ax + By \leqslant b\}$$

$$x = 0.2$$



$\mathcal{N}(P_x)$

P_x and $\mathcal{N}(P_x)$

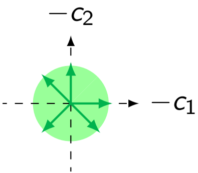


P and P_x

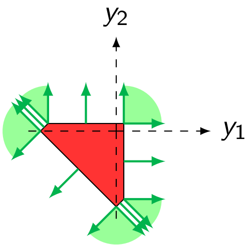
$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x, y) \mid Ax + By \leqslant b\} \quad \text{and} \quad P_x := \{y \mid Ax + By \leqslant b\}$$

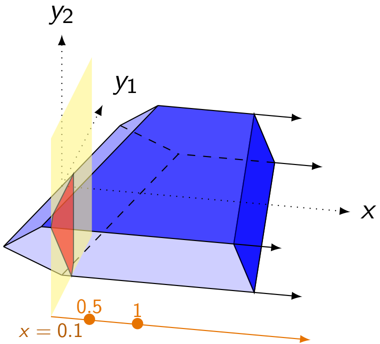
$x = 0.1$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

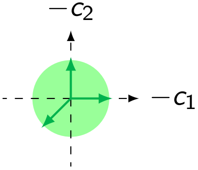


P and P_x

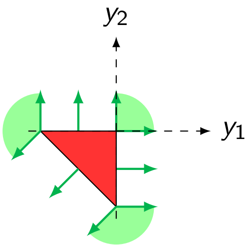
$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x,y) \mid Ax + By \leqslant b\} \quad \text{and} \quad P_x := \{y \mid Ax + By \leqslant b\}$$

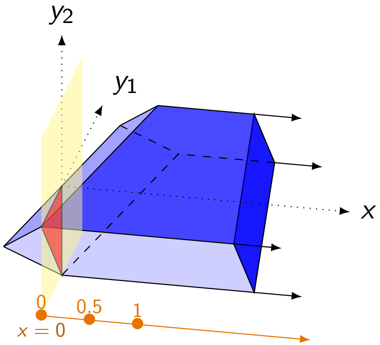
$x = 0$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

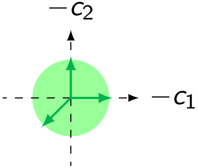


P and P_x

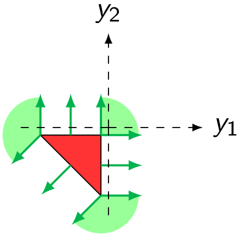
$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x,y) \mid Ax + By \leqslant b\} \quad \text{and} \quad P_x := \{y \mid Ax + By \leqslant b\}$$

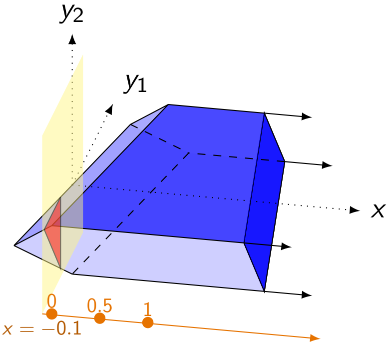
$$x = -0.1$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

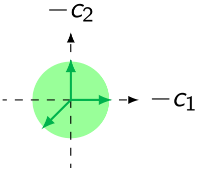


P and P_x

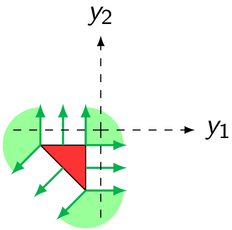
$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x, y) \mid Ax + By \leqslant b\} \quad \text{and} \quad P_x := \{y \mid Ax + By \leqslant b\}$$

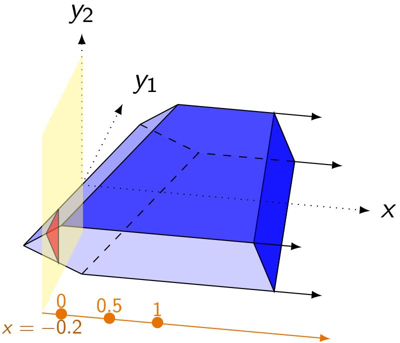
$$x = -0.2$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

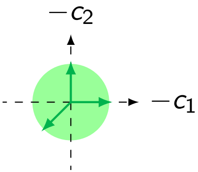


P and P_x

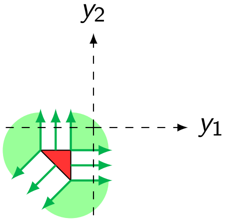
$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x, y) \mid Ax + By \leqslant b\} \quad \text{and} \quad P_x := \{y \mid Ax + By \leqslant b\}$$

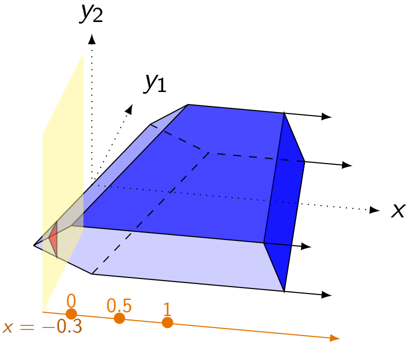
$$x = -0.3$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

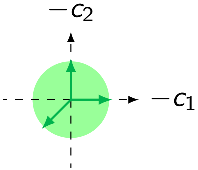


P and P_x

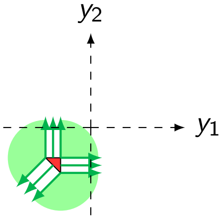
$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x,y) \mid Ax + By \leqslant b\} \quad \text{and} \quad P_x := \{y \mid Ax + By \leqslant b\}$$

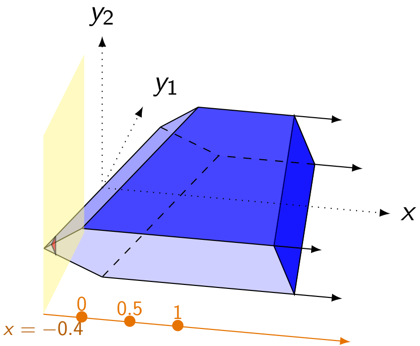
$$x = -0.4$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$



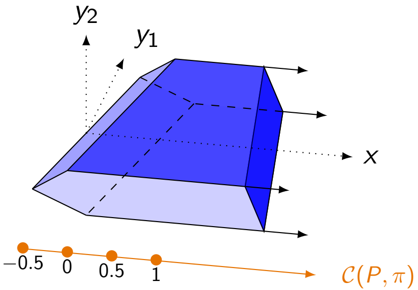
P and P_x

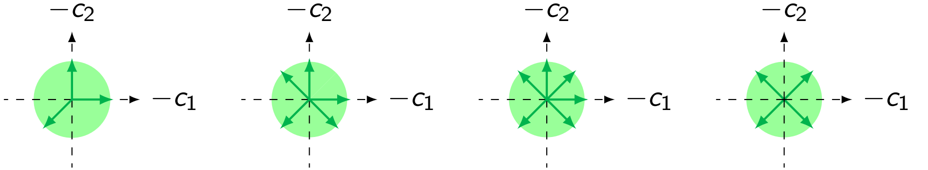
What are the constant regions of $\mathcal{N}(P_x)$?

Lemma

There exists a collection $\mathcal{C}(P,\pi)$ *called the* **chamber complex** *whose relative interior of cells are the constant regions of* $x \to \mathcal{N}(P_x)$.

For $\sigma \in \mathcal{C}(P,\pi)$ *and* $x, x' \in \mathrm{ri}(\sigma)$,
$\mathcal{N}(P_x) = \mathcal{N}(P_{x'}) =: \mathcal{N}_\sigma$





$\mathcal{N}_\sigma$ for $\sigma = [-0.5, 0]$ $\mathcal{N}_\sigma$ for $\sigma = [0, 0.5]$ $\mathcal{N}_\sigma$ for $\sigma = [0.5, 1]$ $\mathcal{N}_\sigma$ for $\sigma = [1, +\infty)$

Chamber complex

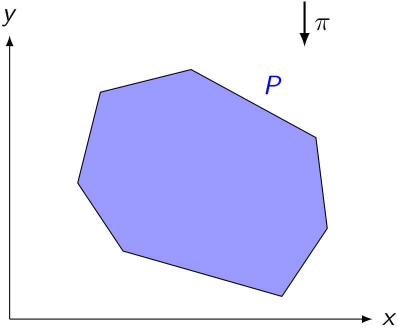
Definition (Billera, Sturmfels 92)

The *chamber complex* $\mathcal{C}(P,\pi)$ of P along π is

$$\mathcal{C}(P,\pi) := \{\sigma_{P,\pi}(x) \mid x \in \pi(P)\}$$

where

$$\sigma_{P,\pi}(x) := \bigcap_{F \in \mathcal{F}(P) \text{ s.t. } x \in \pi(F)} \pi(F)$$



where $\mathcal{F}(P)$ is the set of faces of P and π is the projection $(x,y) \to x$

$$\pi(E) := \{x \in \mathbb{R}^n \mid \exists y \in \mathbb{R}^m,\ (x,y) \in E\}$$

Chamber complex

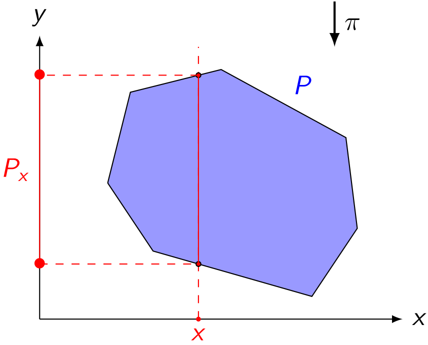
Definition (Billera, Sturmfels 92)

The *chamber complex* $\mathcal{C}(P, \pi)$ of P along π is

$$\mathcal{C}(P,\pi) := \{\sigma_{P,\pi}(x) \mid x \in \pi(P)\}$$

where

$$\sigma_{P,\pi}(x) := \bigcap_{F \in \mathcal{F}(P) \text{ s.t. } x \in \pi(F)} \pi(F)$$



where $\mathcal{F}(P)$ is the set of faces of P and π is the projection $(x, y) \to x$

$$\pi(E) := \{x \in \mathbb{R}^n \mid \exists y \in \mathbb{R}^m,\ (x, y) \in E\}$$

Chamber complex

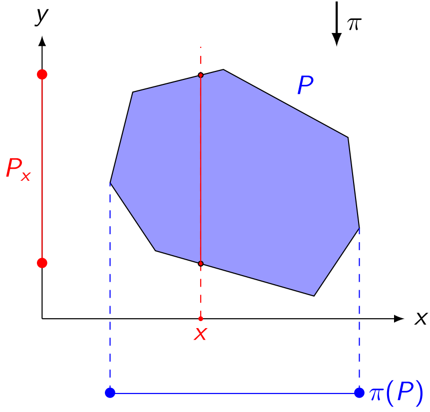
Definition (Billera, Sturmfels 92)

The *chamber complex* $\mathcal{C}(P,\pi)$ of P along π is

$$\mathcal{C}(P,\pi) := \{\sigma_{P,\pi}(x) \mid x \in \pi(P)\}$$

where

$$\sigma_{P,\pi}(x) := \bigcap_{F \in \mathcal{F}(P) \text{ s.t. } x \in \pi(F)} \pi(F)$$



where $\mathcal{F}(P)$ is the set of faces of P and π is the projection $(x,y) \to x$

$$\pi(E) := \{x \in \mathbb{R}^n \mid \exists y \in \mathbb{R}^m,\ (x,y) \in E\}$$

Chamber complex

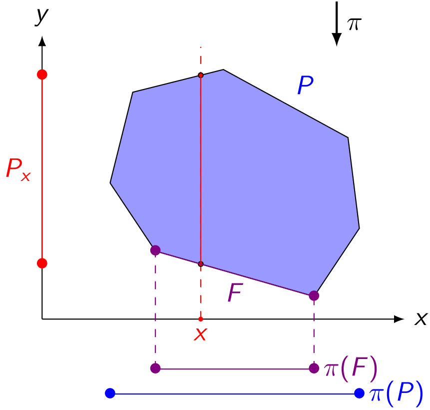
Definition (Billera, Sturmfels 92)

The *chamber complex* $\mathcal{C}(P,\pi)$ of P along π is

$$\mathcal{C}(P,\pi) := \{\sigma_{P,\pi}(x) \mid x \in \pi(P)\}$$

where

$$\sigma_{P,\pi}(x) := \bigcap_{F \in \mathcal{F}(P) \text{ s.t. } x \in \pi(F)} \pi(F)$$



where $\mathcal{F}(P)$ is the set of faces of P and π is the projection $(x,y) \to x$

$$\pi(E) := \{x \in \mathbb{R}^n \mid \exists y \in \mathbb{R}^m,\ (x,y) \in E\}$$

Chamber complex

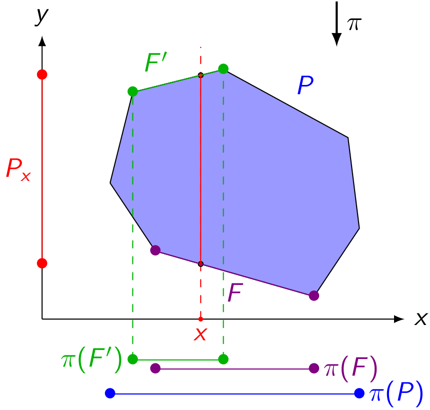
Definition (Billera, Sturmfels 92)

The *chamber complex* $\mathcal{C}(P,\pi)$ of P along π is

$$\mathcal{C}(P,\pi) := \{\sigma_{P,\pi}(x) \mid x \in \pi(P)\}$$

where

$$\sigma_{P,\pi}(x) := \bigcap_{F\in\mathcal{F}(P)\,\text{s.t.}\ x\in\pi(F)} \pi(F)$$



where $\mathcal{F}(P)$ is the set of faces of P and π is the projection $(x,y) \to x$

$$\pi(E) := \{x \in \mathbb{R}^n \mid \exists y \in \mathbb{R}^m,\ (x,y) \in E\}$$

Chamber complex

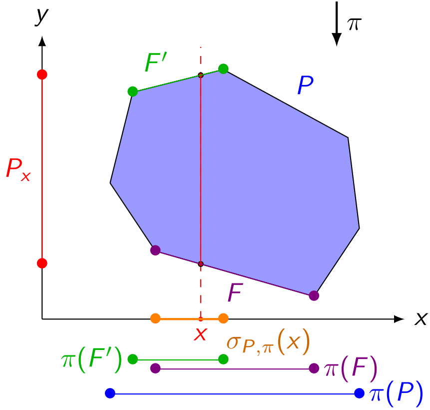
Definition (Billera, Sturmfels 92)

The *chamber complex* $\mathcal{C}(P,\pi)$ of P along π is

$$\mathcal{C}(P,\pi) := \{\sigma_{P,\pi}(x) \mid x \in \pi(P)\}$$

where

$$\sigma_{P,\pi}(x) := \bigcap_{F\in\mathcal{F}(P)\,\text{s.t.}\ x\in\pi(F)} \pi(F)$$



where $\mathcal{F}(P)$ is the set of faces of P and π is the projection $(x,y) \to x$

$$\pi(E) := \{x \in \mathbb{R}^n \mid \exists y \in \mathbb{R}^m,\ (x,y) \in E\}$$

Chamber complex

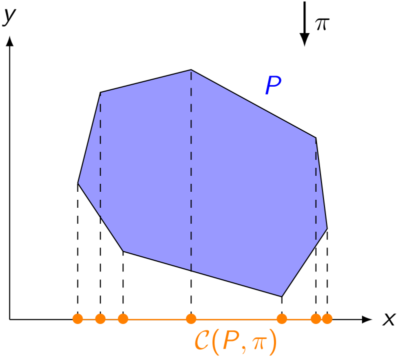
Definition (Billera, Sturmfels 92)

The *chamber complex* $\mathcal{C}(P,\pi)$ of P along π is

$$\mathcal{C}(P,\pi) := \{\sigma_{P,\pi}(x) \mid x \in \pi(P)\}$$

where

$$\sigma_{P,\pi}(x) := \bigcap_{F \in \mathcal{F}(P) \text{ s.t. } x \in \pi(F)} \pi(F)$$

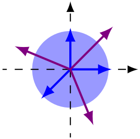


where $\mathcal{F}(P)$ is the set of faces of P and π is the projection $(x,y) \to x$

$$\pi(E) := \{x \in \mathbb{R}^n \mid \exists y \in \mathbb{R}^m,\ (x,y) \in E\}$$

Common Refinement of Normal Fans

We can quantize $\mathbf{c}$ on each chamber.



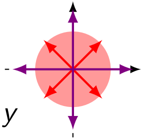
$\mathcal{N}_\sigma$ and $\check{\mathbf{c}}$

For all $x \in \mathrm{ri}(\sigma)$,

$$V(x) = \sum_{N \in \mathcal{N}_\sigma} p_N \min_{y \in P_x} \check{c}_N^\top y$$

For all $x \in \mathrm{ri}(\tau)$,

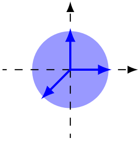
$$V(x) = \sum_{N \in \mathcal{N}_\tau} p_N \min_{y \in P_x} \check{c}_N^\top y$$



$\mathcal{N}_\tau$ and $\check{\mathbf{c}}$

Common Refinement of Normal Fans

We can quantize **c** on each chamber.



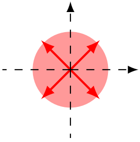
$\mathcal{N}_\sigma$

For all $x \in \mathrm{ri}(\sigma)$,

$$V(x) = \sum_{N \in \mathcal{N}_\sigma} p_N \min_{y \in P_x} \check{c}_N^\top y$$

For all $x \in \mathrm{ri}(\tau)$,

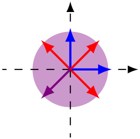
$$V(x) = \sum_{N \in \mathcal{N}_\tau} p_N \min_{y \in P_x} \check{c}_N^\top y$$



$\mathcal{N}_\tau$

We take the *common refinement*:

$$\mathcal{R} := \mathcal{N}_\sigma \wedge \mathcal{N}_\tau = \{N \cap N' \mid N \in \mathcal{N}_\sigma, N' \in \mathcal{N}_\tau\}$$



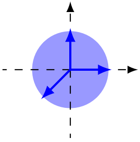
$\mathcal{N}_\sigma \wedge \mathcal{N}_\tau$

For all $x \in \mathrm{ri}(\sigma) \cup \mathrm{ri}(\tau)$,

$$V(x) = \sum_{N \in \mathcal{N}_\sigma \wedge \mathcal{N}_\tau} p_N \min_{y \in P_x} \check{c}_N^\top y$$

Common Refinement of Normal Fans

We can quantize $\mathbf{c}$ on each chamber.



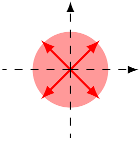
$\mathcal{N}_\sigma$

For all $x \in \mathrm{ri}(\sigma)$,

$$V(x) = \sum_{N \in \mathcal{N}_\sigma} p_N \min_{y \in P_x} \check{c}_N^\top y$$

For all $x \in \mathrm{ri}(\tau)$,

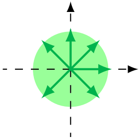
$$V(x) = \sum_{N \in \mathcal{N}_\tau} p_N \min_{y \in P_x} \check{c}_N^\top y$$



$\mathcal{N}_\tau$

We take the *common refinement*:

$$\mathcal{R} := \mathcal{N}_\sigma \wedge \mathcal{N}_\tau = \{N \cap N' \mid N \in \mathcal{N}_\sigma, N' \in \mathcal{N}_\tau\}$$



$\mathcal{R}$

For all $x \in \mathrm{ri}(\sigma) \cup \mathrm{ri}(\tau)$,

$$V(x) = \sum_{N \in \mathcal{R}} p_N \min_{y \in P_x} \check{c}_N^\top y$$

General cost $\mathbf{c}$ is equivalent to discrete cost $\check{\mathbf{c}}$ for all x

Theorem (Quantization of the cost distribution)

Let $\mathcal{R} = \bigwedge_{\sigma \in \mathcal{C}(P,\pi)} -\mathcal{N}_\sigma$, *then* **for all** $x \in \mathbb{R}^n$

$$V(x) = \sum_{R \in \mathcal{R}} \check{p}_R \min_{y \in P_x} \check{c}_R^\top y$$

where $\check{p}_R := \mathbb{P}\big[\mathbf{c} \in \mathrm{ri}(R)\big]$ *and* $\check{c}_R := \mathbb{E}\big[\mathbf{c} \,|\, \mathbf{c} \in \mathrm{ri}(R)\big]$
Moreover, for all distributions of $\mathbf{c}$,
V *is affine on each cell of the chamber complex* $\mathcal{C}(P, \pi)$.

Bonus: This quantization method works for *every distribution of* $\mathbf{c}$!

General cost $\mathbf{c}$ is equivalent to discrete cost $\check{\mathbf{c}}$ for all x

Theorem (Quantization of the cost distribution)

Let $\mathcal{R} = \bigwedge_{\sigma \in \mathcal{C}(P,\pi)} -\mathcal{N}_\sigma$, *then* **for all** $x \in \mathbb{R}^n$

$$V(x) = \sum_{R \in \mathcal{R}} \check{p}_R \min_{y \in P_x} \check{c}_R^\top y$$

where $\check{p}_R := \mathbb{P}\big[\mathbf{c} \in \mathrm{ri}(R)\big]$ *and* $\check{c}_R := \mathbb{E}\big[\mathbf{c} \,|\, \mathbf{c} \in \mathrm{ri}(R)\big]$
Moreover, for all distributions of $\mathbf{c}$,
V *is affine on each cell of the chamber complex* $\mathcal{C}(P,\pi)$.

Bonus: This quantization method works for *every distribution of* $\mathbf{c}$!

Extension to multistage and stochastic constraints

Theorem

All results generalize to stochastic constraints with finite support and multistage

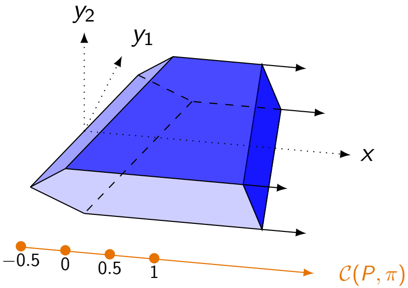
- $\rightsquigarrow$ *The regions where $(V_t)_t$ is affine do not depend on the $(\mathbf{c}_t)_t$*
- $\rightsquigarrow$ *We have an exact discretization method working for all $(\mathbf{c}_t)_t$*

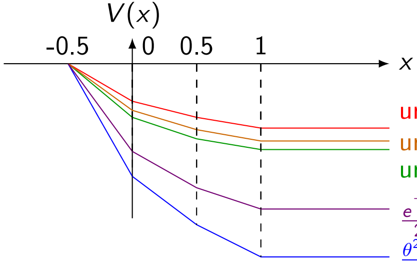
Idea of the proof :
Iterated chamber complexes

$$\mathcal{P}_{t,\xi} := \mathcal{C}\big(\mathbb{R}^{n_t} \times \mathcal{P}_{t+1} \wedge \mathcal{F}\big(P_t(\xi)\big), \pi_{x_{t-1}}^{x_{t-1},x_t}\big)$$
$$\mathcal{P}_t := \bigwedge_{\xi_t \in \mathrm{supp}\, \xi_t} \mathcal{P}_{t,\xi}$$

Explicit computation of the example

$$V(x) = \mathbb{E}\left[\begin{array}{ll} \min\limits_{y\in\mathbb{R}^2} & \mathbf{c}^\top y \\ \text{s.t.} & \|y\|_1 \leqslant 1 \\ & y_1 \leqslant x \\ & y_2 \leqslant x \end{array}\right]$$





Different distributions of $\mathbf{c}$:

uniform on norm 1 ball

uniform on norm 2 ball

uniform on norm ∞ ball

$\dfrac{e^{-\frac{\|c\|_2^2}{2\gamma^2}}}{2\pi\gamma^2}dc$

$\dfrac{\theta^2 e^{-\theta\|c\|_1}}{4}dc$

Explicit formulas for usual distributions

Distribution	Uniform on polytope	Exponential	Gaussian
$d\mathbb{P}(c)$	$\frac{\mathbb{1}_{c\in Q}}{\mathrm{Vol}_d(Q)} d\mathcal{L}_{\mathrm{Aff}(Q)}(c)$	$\frac{e^{\theta^\top c}\mathbb{1}_{c\in K}}{\Phi_K(\theta)} d\mathcal{L}_{\mathrm{Aff}(K)}c$	$\frac{e^{-\frac{1}{2}c^\top M^{-2}c}}{(2\pi)^{\frac{m}{2}}\det M} dc$
Support	Polytope : Q	Cone : K	$\mathbb{R}^m$
$\mathbb{P}[\mathbf{c}\in S]$	$\dfrac{\mathrm{Vol}_d(S)}{\mathrm{Vol}_d(Q)}$	$\dfrac{\lvert\det(\mathrm{Ray}(S))\rvert}{\Phi_K(\theta)} \prod_{r\in\mathrm{Ray}(S)} \dfrac{1}{-r^\top\theta}$	$\mathrm{Ang}\left(M^{-1}S\right)$
$\mathbb{E}\left[\mathbf{c}\mid\mathbf{c}\in S\right]$	$\frac{1}{d}\sum_{v\in\mathrm{Vert}(S)} v$	$\left(\sum_{r\in\mathrm{Ray}(S)}\frac{-r_i}{r^\top\theta}\right)_{i\in[m]}$	$\frac{\sqrt{2}\Gamma(\frac{m+1}{2})}{\Gamma(\frac{m}{2})} M\,\mathrm{Ctr}\left(S\cap\mathbb{S}_{m-1}\right)$

These formulas are valid for S full dimensional simplex or simplicial cone.

Contents

Dual problem

$$V(x) := \mathbb{E}\begin{bmatrix} \inf\limits_{y} & \mathbf{c}^\top y \\ \text{s.t.} & Ax + By \leqslant b \end{bmatrix} = \mathbb{E}\big[\inf_{y \in P_x} \mathbf{c}^\top y\big]$$

where $P_x = \{x \mid Ax + By \leqslant b\}$

$$V(x) := \mathbb{E}\begin{bmatrix} \sup\limits_{\mu} & (Ax - b)^\top \mu \\ \text{s.t.} & B^\top \mu + \mathbf{c} = 0 \\ & \mu \geqslant 0 \end{bmatrix} = \mathbb{E}\big[\sup_{\mu \in D_{\mathbf{c}}} (Ax - b)^\top \mu\big]$$

where $D_c = \{\mu \mid B^\top \mu + c = 0, \mu \geqslant 0\}$

Fiber Polyhedron

Minkowski sum :

$$E + F = \{x + x' \,|x \in E, x' \in F\}$$

Definition

The *fiber polyhedron* E of the bundle $(D_c)_{c\in \mathrm{supp}(\mathbf{c})}$ is the Minkowsky integral of all the fiber at c when c varies according to its probability distribution:

$$E := \int D_c \mathbb{P}(dc) = \Big\{ \int \mu(c)\mathbb{P}(dc) \,|\, \mu(c) \in D_c \quad a.s.,\ \mu \in L_\infty(\mathbb{R}^m, \mathbb{R}^l) \Big\}$$

$$V(x) = \mathbb{E}\Big[\sup_{\mu \in D_{\mathbf{c}}} (Ax - b)^\top \mu \Big]$$

$$= \begin{cases} \sup_{\mu(\cdot)} & (Ax-b)^\top \mathbb{E}\big[\mu(\mathbf{c})\big] \\ \text{s.t.} & \mu(\mathbf{c}) \in D_{\mathbf{c}} \ \text{ a.s.} \end{cases}$$

Fiber Polyhedron

Minkowski sum :

$$E + F = \{x + x' \,|x \in E, x' \in F\}$$

Definition

The *fiber polyhedron* E of the bundle $(D_c)_{c \in \mathrm{supp}(\mathbf{c})}$ is the Minkowsky integral of all the fiber at c when c varies according to its probability distribution:

$$E := \int D_c \mathbb{P}(dc) = \Big\{ \int \mu(c)\mathbb{P}(dc) \,|\, \mu(c) \in D_c \quad a.s., \ \mu \in L_\infty(\mathbb{R}^m, \mathbb{R}^l) \Big\}$$

$$\begin{aligned} V(x) &= \mathbb{E}\big[\sup_{\mu \in D_\mathbf{c}} (Ax - b)^\top \mu \big] \\ &= \begin{cases} \sup_{\mu(\cdot)} & (Ax-b)^\top \mathbb{E}\left[\mu(\mathbf{c})\right] \\ \text{s.t.} & \mu(\mathbf{c}) \in D_\mathbf{c} \ \text{ a.s.} \end{cases} \end{aligned}$$

Fiber Polyhedron

Minkowski sum :

$$E + F = \{x + x' \,|x \in E, x' \in F\}$$

Definition

The *fiber polyhedron* E of the bundle $(D_c)_{c \in \text{supp}(\mathbf{c})}$ is the Minkowsky integral of all the fiber at c when c varies according to its probability distribution:

$$E := \int D_c \mathbb{P}(dc) = \Big\{ \int \mu(c)\mathbb{P}(dc) \,|\, \mu(c) \in D_c \quad a.s., \; \mu \in L_\infty(\mathbb{R}^m, \mathbb{R}^l) \Big\}$$

$$V(x) = \mathbb{E}\big[\sup_{\mu \in D_\mathbf{c}} (Ax - b)^\top \mu \big]$$

$$= \begin{cases} \sup_{\mu(\cdot)} & (Ax - b)^\top \mathbb{E}\big[\mu(\mathbf{c})\big] \\ \text{s.t.} & \mathbb{E}\big[\mu(\mathbf{c})\big] \in E \end{cases}$$

Fiber Polyhedron

Minkowski sum :

$$E + F = \{x + x' \,|x \in E, x' \in F\}$$

Definition

The *fiber polyhedron* E of the bundle $(D_c)_{c \in \mathrm{supp}(\mathbf{c})}$ is the Minkowsky integral of all the fiber at c when c varies according to its probability distribution:

$$E := \int D_c \mathbb{P}(dc) = \Big\{ \int \mu(c)\mathbb{P}(dc) \,|\, \mu(c) \in D_c \quad a.s.,\ \mu \in L_\infty(\mathbb{R}^m, \mathbb{R}^l) \Big\}$$

$$\begin{aligned} V(x) &= \mathbb{E}\big[\sup_{\mu \in D_\mathbf{c}} (Ax - b)^\top \mu \big] \\ &= \begin{cases} \sup_{\mu(\cdot)} & (Ax - b)^\top \mathbb{E}\big[\mu(\mathbf{c})\big] \\ \text{s.t.} & \mathbb{E}\big[\mu(\mathbf{c})\big] \in E \end{cases} \\ &= \sup_{\lambda \in E} (Ax - b)^\top \lambda \\ &= \sigma_E(Ax - b) \end{aligned}$$

The Fiber Polyhedron is a finite Minkowski sum

Theorem

There exists a chamber complex $\mathcal{R}$ depending on A such that

$$E = \int D_c \mathbb{P}(dc) = \sum_{R \in \mathcal{R}} \check{p}_R D_{\check{c}_R}$$

where $\check{p}_R := \mathbb{P}\big[\mathbf{c} \in \mathrm{ri}(R)\big]$ *and* $\check{c}_R := \mathbb{E}\big[\mathbf{c} \,|\, \mathbf{c} \in \mathrm{ri}(R)\big]$.

Alternative proof of the quantization result

$$V(x) = \sigma_E(Ax - b) = \sum_{R \in \mathcal{R}} \check{p}_R \sigma_{D_{\check{c}_R}}(Ax - b) = \sum_{R \in \mathcal{R}} \check{p}_R \min_{y \in P_x} \check{c}_R^\top y$$

The Fiber Polyhedron is a finite Minkowski sum

Theorem

There exists a chamber complex $\mathcal{R}$ depending on A such that

$$\mathbf{E} = \int D_c \mathbb{P}(dc) = \sum_{R \in \mathcal{R}} \check{p}_R D_{\check{c}_R}$$

where $\check{p}_R := \mathbb{P}\big[\mathbf{c} \in \mathrm{ri}(R)\big]$ *and* $\check{c}_R := \mathbb{E}\big[\mathbf{c} \,|\, \mathbf{c} \in \mathrm{ri}(R)\big]$.

Alternative proof of the quantization result

$$V(x) = \sigma_E(Ax - b) = \sum_{R \in \mathcal{R}} \check{p}_R \sigma_{D_{\check{c}_R}}(Ax - b) = \sum_{R \in \mathcal{R}} \check{p}_R \min_{y \in P_x} \check{c}_R^\top y$$

Nested Fiber Polyhedra for Multistage

$$V_t(x_{t-1}) = \mathbb{E}\left[\begin{array}{l} \min\limits_{x_t \in \mathbb{R}^{n_t}} \mathbf{c}_t^\top x_t + V_{t+1}(x_t) \\ \text{s.t. } A_t x_t + B_t x_{t-1} \leqslant b_t \end{array}\right]$$

Definition

We define by induction the following nested fiber polyhedra

$$\begin{aligned} D_{t,c_t} &:= \{\mu_t | \mu_t \geqslant 0, A_t^\top \mu_t + c_t = 0\} && \forall t \in [T] \\ F_{T,c_T} &:= D_{T,c_T} \\ E_t &:= \mathbb{E}\left[F_{t,\mathbf{c}_t}\right] && \forall t \in [T] \\ F_{t,c_t} &:= \{(\mu_t, \lambda_{[t+1:T]}) \,|\, \mu_t \in D_{t,c_t + B_{t+1}^\top \lambda_{t+1}},\ \lambda_{[t+1:T]} \in E_{t+1}\} && \forall t \in [T-1] \end{aligned}$$

$$V_t(x_{t-1}) = \sigma_{E_t}(B_t x_{t-1} - b_t, -b_{[t+1:T]})$$

Nested Fiber Polyhedra for Multistage

$$V_t(x_{t-1}) = \mathbb{E}\left[\begin{array}{l} \min_{x_t \in \mathbb{R}^{n_t}} \mathbf{c}_t^\top x_t + V_{t+1}(x_t) \\ \text{s.t. } A_t x_t + B_t x_{t-1} \leqslant b_t \end{array}\right]$$

Definition

We define by induction the following nested fiber polyhedra

$$\begin{aligned} D_{t,c_t} &:= \{\mu_t | \mu_t \geqslant 0, A_t^\top \mu_t + c_t = 0\} && \forall t \in [T] \\ F_{T,c_T} &:= D_{T,c_T} \\ E_t &:= \mathbb{E}\big[F_{t,\mathbf{c}_t}\big] && \forall t \in [T] \\ F_{t,c_t} &:= \{(\mu_t, \lambda_{[t+1:T]}) \,|\, \mu_t \in D_{t,c_t + B_{t+1}^\top \lambda_{t+1}},\ \lambda_{[t+1:T]} \in E_{t+1}\} && \forall t \in [T-1] \end{aligned}$$

$$V_t(x_{t-1}) = \sigma_{E_t}(B_t x_{t-1} - b_t, -b_{[t+1:T]})$$

2-time scale MSLP reduced to Quadratic Problem

We consider a MSLP where the dynamic depends on parameters p we have to optimize

$$
\begin{aligned}
\min_{p\in\mathbb{R}^m,(\mathbf{x}_t)\in\mathbb{R}^{n_t}} \quad & q^\top p + \mathbb{E}\big[\sum_{t=1}^{T} \mathbf{c}_t^\top \mathbf{x}_t\big] & \\
\text{s.t.} \quad & Dp \leqslant d & \\
& A_t\mathbf{x}_t + B_t\mathbf{x}_{t-1} + C_t p \leqslant h_t \quad \text{a.s.} & \forall t \in [T] \\
& \mathbf{x}_t \prec \sigma(\mathbf{c}_1, \cdots, \mathbf{c}_t) & \forall t \in [T]
\end{aligned}
$$

If we know the fiber polyhedron, it reduces to a finite dimensional quadratic problem

$$
\begin{aligned}
\sup_{p\in\mathbb{R}^m,(\lambda_t)_{t\in[T]}} \quad & -q^\top p + \sum_{t=1}^{T}(C_t p - h_t)^\top \lambda_t \\
\text{s.t.} \quad & Dp \leqslant d \\
& (\lambda_1, \cdots, \lambda_T) \in E_1
\end{aligned}
$$

Contents

Earlier and new complexity results

Volume of a polytope

$$\mathrm{Vol}\left(\{z \in \mathbb{R}^d \mid Az \leqslant b\}\right) \text{ or}$$
$$\mathrm{Vol}\left(\,\mathrm{Conv}(v_1, \cdots, v_n)\right)$$

- $\sharp P$-complete: Dyer and Frieze (1988)
- Polynomial for fixed dimension d: Barvinok (1994)

2-stage linear problem

$$\min_{x \in \mathbb{R}^n} c_0^\top x + \mathbb{I}_{Ax \leqslant b} + \mathbb{E}\Big[\min_{y \in \mathbb{R}^m} \mathbf{c}^\top y + \mathbb{I}_{\mathbf{T}x + \mathbf{W}y \leqslant \mathbf{h}}\Big]$$

- $\sharp P$-hard: Hanasusanto, Kuhn and Wiesemann (2016)
- Polynomial for fixed m ?

Earlier and new complexity results

Volume of a polytope

$$\mathrm{Vol}\left(\{z \in \mathbb{R}^d \mid Az \leqslant b\}\right) \text{ or}$$
$$\mathrm{Vol}\left(\mathrm{Conv}(v_1, \cdots, v_n)\right)$$

- $\sharp P$-complete:
  Dyer and Frieze (1988)
- Polynomial for fixed dimension d:
  Barvinok (1994)

2-stage linear problem

$$\min_{x \in \mathbb{R}^n} c_0^\top x + \mathbb{I}_{Ax \leqslant b} + \mathbb{E}\Big[\min_{y \in \mathbb{R}^m} \mathbf{c}^\top y + \mathbb{I}_{\mathbf{T}x + \mathbf{W}y \leqslant \mathbf{h}}\Big]$$

- $\sharp P$-hard: Hanasusanto, Kuhn and Wiesemann (2016)
- Polynomial for fixed m:
  FGL (2020)

Complexity result multistage

We can generalize to multistage by fixing several dimensions and the horizon.

Theorem (MSLP is polynomial for fixed dimensions)

Assume that $T \geqslant 3, n_2, \ldots, n_T$, $\sharp\big(\operatorname{supp}(\mathbf{A}_2, \mathbf{B}_2, \mathbf{b}_2)\big)$, $\cdots$, $\sharp\big(\operatorname{supp}(\mathbf{A}_T, \mathbf{B}_T, \mathbf{b}_T)\big)$ *are fixed integers*

and for all $t \in [T]$, $\mathbf{c}_t$ *conditionally to* $\{(\mathbf{A}_t, \mathbf{B}_t, \mathbf{b}_t) = (A, B, b)\}$ *is easily computable.*

Then, we can solve MSLP in polynomial time.

Conclusion

- MSLP with arbitrary cost distribution can be exactly discretized;
- new algebraic insights on the polyhedral structure of MSLP;
- analytical formulas for some usual distributions;
- fixed-parameter versions of 2SLP and MSLP are polynomial time.

Perspectives

$\rightsquigarrow$ New algorithms from the algebraic structure

$\rightsquigarrow$ Sensibility analysis to the distribution, link with nested distance;

$\rightsquigarrow$ Extend to integer stochastic problems;

$\rightsquigarrow$ Distributionnally robust optimization.

Conclusion

- MSLP with arbitrary cost distribution can be exactly discretized;
- new algebraic insights on the polyhedral structure of MSLP;
- analytical formulas for some usual distributions;
- fixed-parameter versions of 2SLP and MSLP are polynomial time.

Perspectives

$\rightsquigarrow$ New algorithms from the algebraic structure

$\rightsquigarrow$ Sensibility analysis to the distribution, link with nested distance;

$\rightsquigarrow$ Extend to integer stochastic problems;

$\rightsquigarrow$ Distributionnally robust optimization.

Conclusion

- MSLP with arbitrary cost distribution can be exactly discretized;
- new algebraic insights on the polyhedral structure of MSLP;
- analytical formulas for some usual distributions;
- fixed-parameter versions of 2SLP and MSLP are polynomial time.

Perspectives

$\rightsquigarrow$ New algorithms from the algebraic structure

$\rightsquigarrow$ Sensibility analysis to the distribution, link with nested distance;

$\rightsquigarrow$ Extend to integer stochastic problems;

$\rightsquigarrow$ Distributionnally robust optimization.

Conclusion

- MSLP with arbitrary cost distribution can be exactly discretized;
- new algebraic insights on the polyhedral structure of MSLP;
- analytical formulas for some usual distributions;
- fixed-parameter versions of 2SLP and MSLP are polynomial time.

Perspectives

$\rightsquigarrow$ New algorithms from the algebraic structure

$\rightsquigarrow$ Sensibility analysis to the distribution, link with nested distance;

$\rightsquigarrow$ Extend to integer stochastic problems;

$\rightsquigarrow$ Distributionnally robust optimization.

Conclusion

- MSLP with arbitrary cost distribution can be exactly discretized;
- new algebraic insights on the polyhedral structure of MSLP;
- analytical formulas for some usual distributions;
- fixed-parameter versions of 2SLP and MSLP are polynomial time.

Perspectives

$\rightsquigarrow$ New algorithms from the algebraic structure

$\rightsquigarrow$ Sensibility analysis to the distribution, link with nested distance;

$\rightsquigarrow$ Extend to integer stochastic problems;

$\rightsquigarrow$ Distributionnally robust optimization.

Conclusion

- MSLP with arbitrary cost distribution can be exactly discretized;
- new algebraic insights on the polyhedral structure of MSLP;
- analytical formulas for some usual distributions;
- fixed-parameter versions of 2SLP and MSLP are polynomial time.

Perspectives

$\rightsquigarrow$ New algorithms from the algebraic structure

$\rightsquigarrow$ Sensibility analysis to the distribution, link with nested distance;

$\rightsquigarrow$ Extend to integer stochastic problems;

$\rightsquigarrow$ Distributionnally robust optimization.

Conclusion

- MSLP with arbitrary cost distribution can be exactly discretized;
- new algebraic insights on the polyhedral structure of MSLP;
- analytical formulas for some usual distributions;
- fixed-parameter versions of 2SLP and MSLP are polynomial time.

Perspectives

$\rightsquigarrow$ New algorithms from the algebraic structure

$\rightsquigarrow$ Sensibility analysis to the distribution, link with nested distance;

$\rightsquigarrow$ Extend to integer stochastic problems;

$\rightsquigarrow$ Distributionnally robust optimization.

References

[1] Louis J Billera and Bernd Sturmfels. Fiber polytopes. *Annals of Mathematics*, pages 527–549, 1992.

[2] Jesús A De Loera, Jörg Rambau, and Francisco Santos. *Triangulations Structures for algorithms and applications*. Springer, 2010.

[3] Martin Grötschel, László Lovász, and Alexander Schrijver. *Geometric algorithms and combinatorial optimization*, volume 2. Springer Science & Business Media, 2012.

[4] Grani A Hanasusanto, Daniel Kuhn, and Wolfram Wiesemann. A comment on "computational complexity of stochastic programming problems". *Mathematical Programming*, 159(1-2):557–569, 2016.

[5] Alexander Shapiro, Darinka Dentcheva, and Andrzej Ruszczyński. *Lectures on stochastic programming: modeling and theory*. SIAM, 2014.

Thank you for listening ! Any question ?

- Maël Forcier, Stéphane Gaubert and Vincent Leclère, *The Polyhedral Structure and Complexity of Multistage Stochastic Linear Problem with General Cost Distribution*, `https://hal.archives-ouvertes.fr/hal-02929361`.